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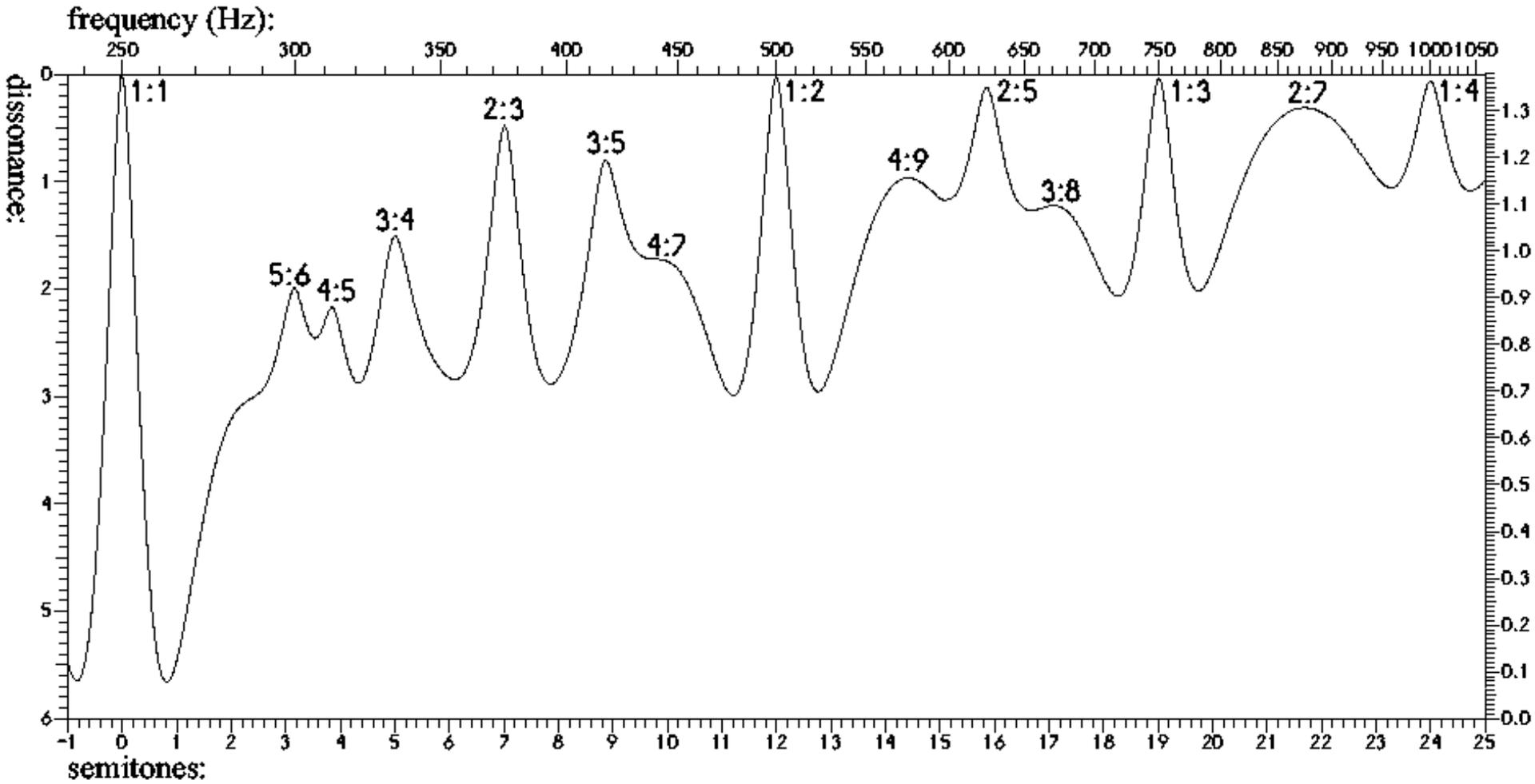
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On the Harmonic Rationalization of
Pitch Intervals of Known Size

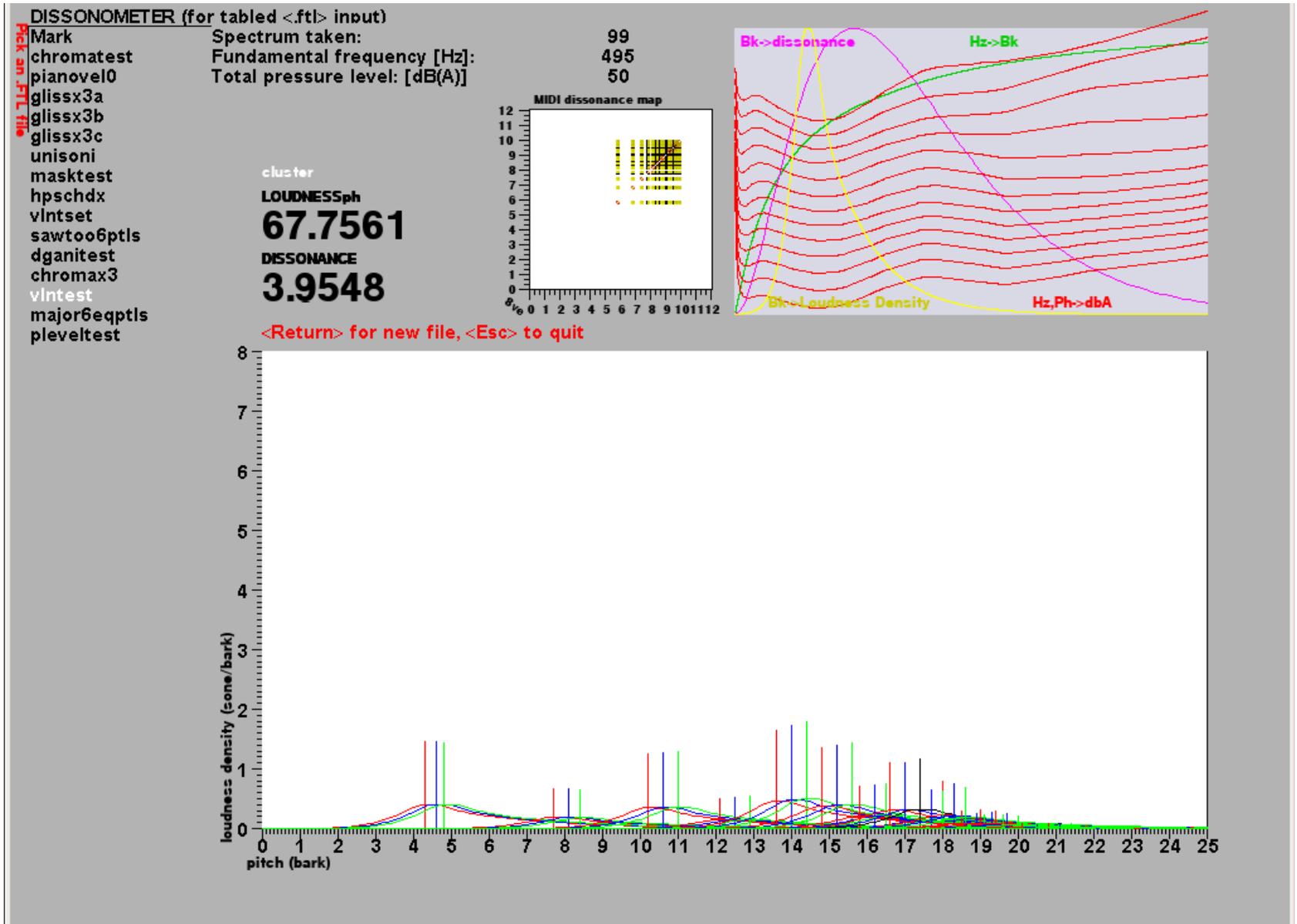
Hindemith: twelve chromatic pitches
in order of decreasing consonance



Sensory Dissonance Curve after Plomp and Levelt



Screenshot of *Dissonometer*



Two Euler functions

Euler's	$n:$	2	3	4	5	6	7	8	9	10	11	12
Totient function	$\varphi(n) = n \prod (1 - 1/p)$:	1	2	2	4	2	6	4	6	4	10	4
Gradus suavitatus	$\Gamma(n) = 1 + \Sigma(p-1)$:	2	3	3	5	4	7	4	5	6	11	5

Formula for Indigestibility ξ
of a whole number N
with tabulated values

$$\xi(N) = 2 \sum_{r=1}^{\infty} \left(\frac{n_r (p_r - 1)^2}{p_r} \right)$$

whereby:

$$1. N = \prod_{r=1}^{\infty} p_r^{n_r}$$

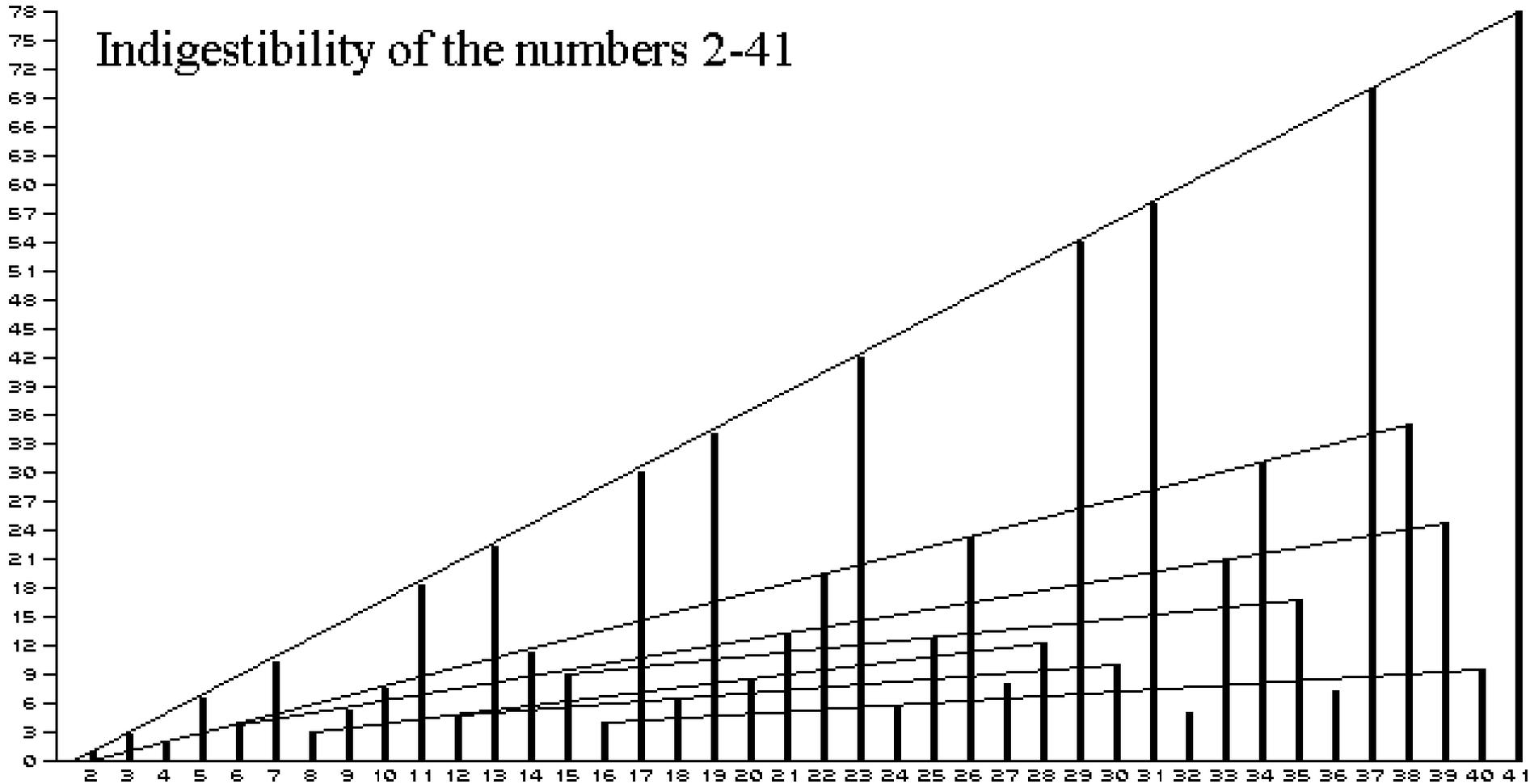
2. $N, n, p \in$ natural numbers

3. $p \in$ prime numbers

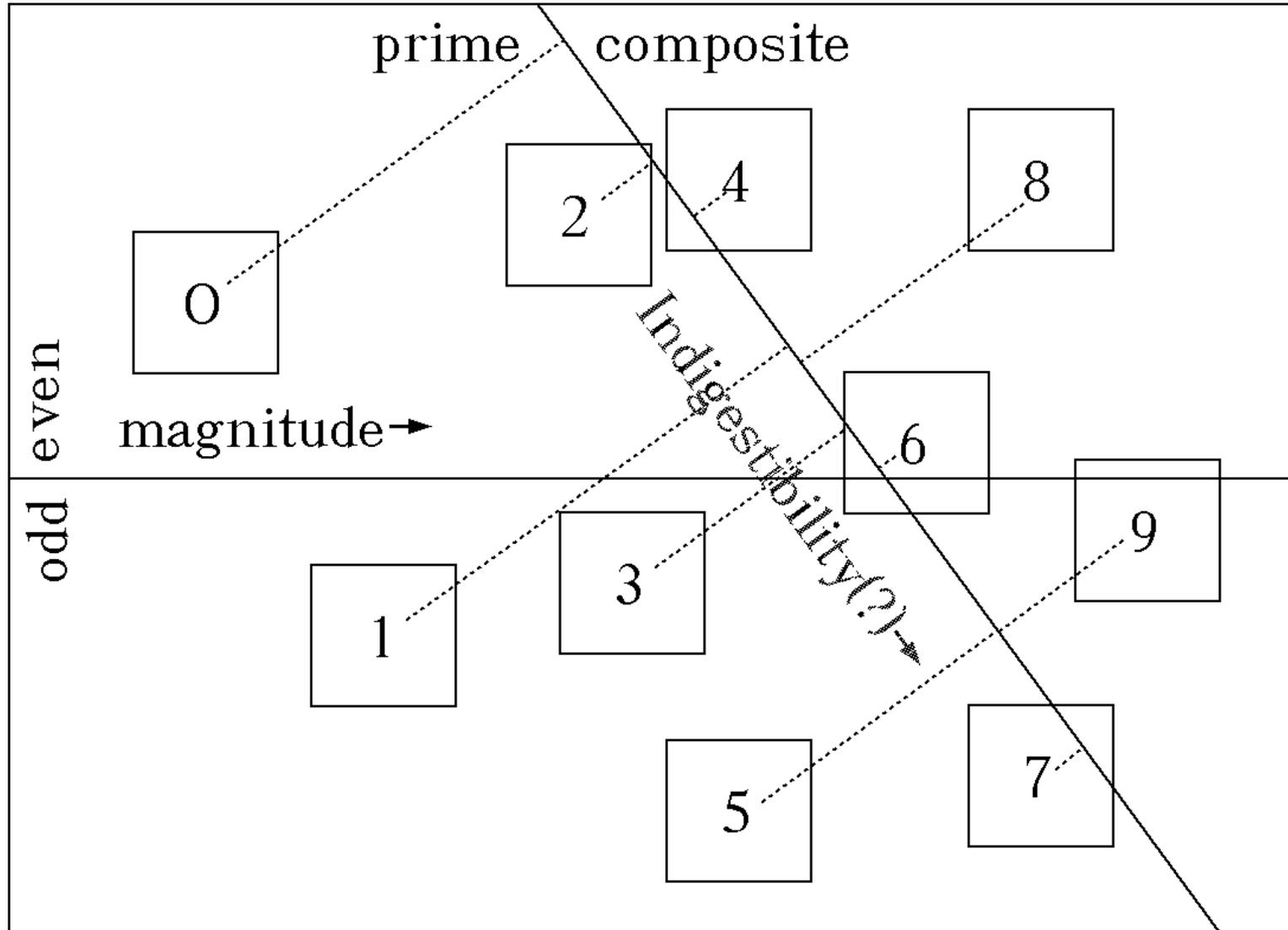
N	$\xi(N)$
1	0.000000
2	1.000000
3	2.666667
4	2.000000
5	6.400000
6	3.666667
7	10.285714
8	3.000000
9	5.333333
10	7.400000
11	18.181818
12	4.666667
13	22.153846
14	11.285714
15	9.066667
16	4.000000

Graphical depiction of Indigestibility

Indigestibility of the numbers 2-41



Stanford Digit Similarity Test (1978)



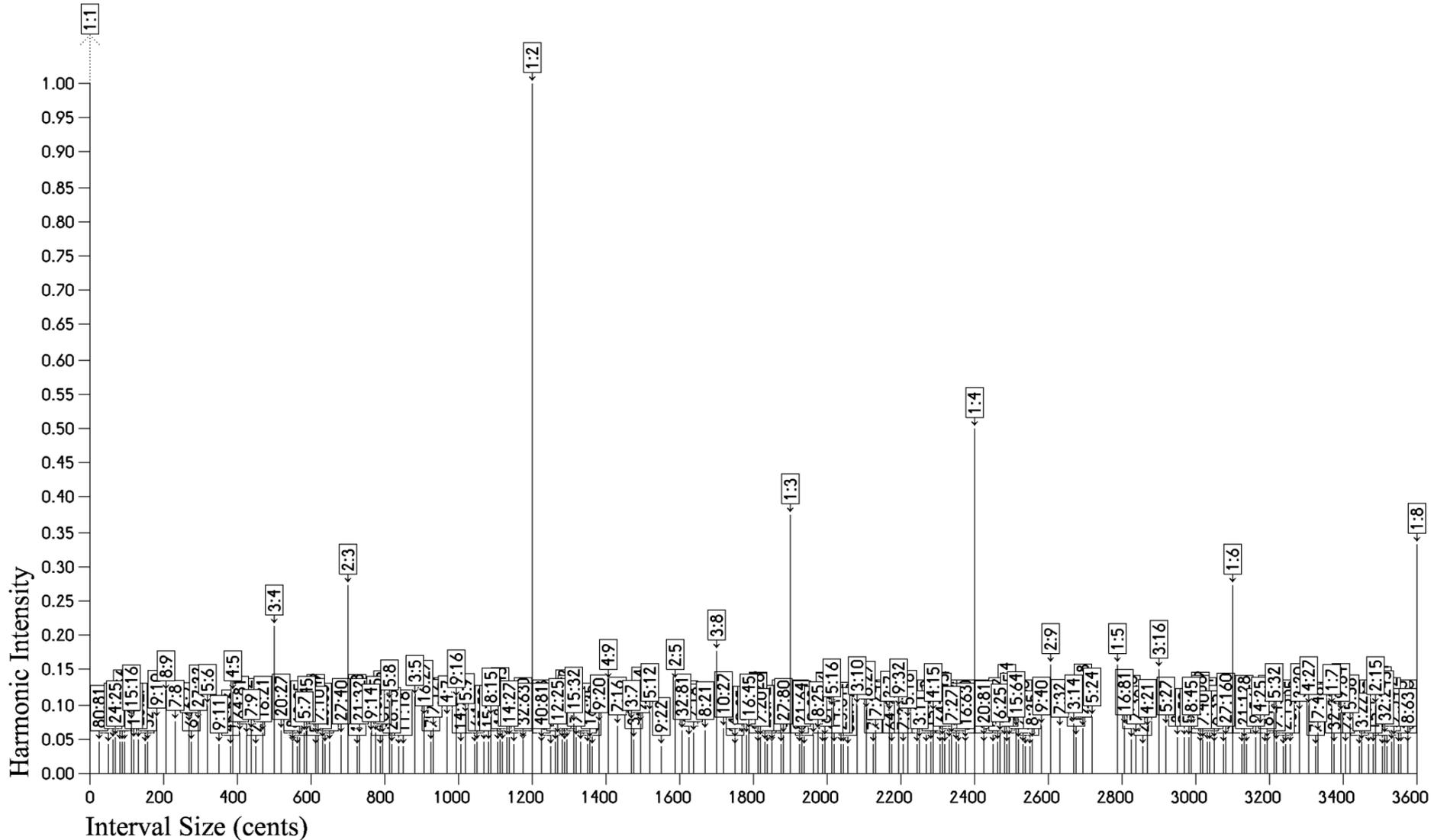
Formula for the
Harmonicity \mathcal{H} of a
Pitch Interval with
frequency ratio $P:Q$
($Q>P$)
and tabulated values

$$\mathcal{H}(P,Q) = \frac{\text{sgn}(\xi(Q) - \xi(P))}{\xi(P) + \xi(Q)}$$

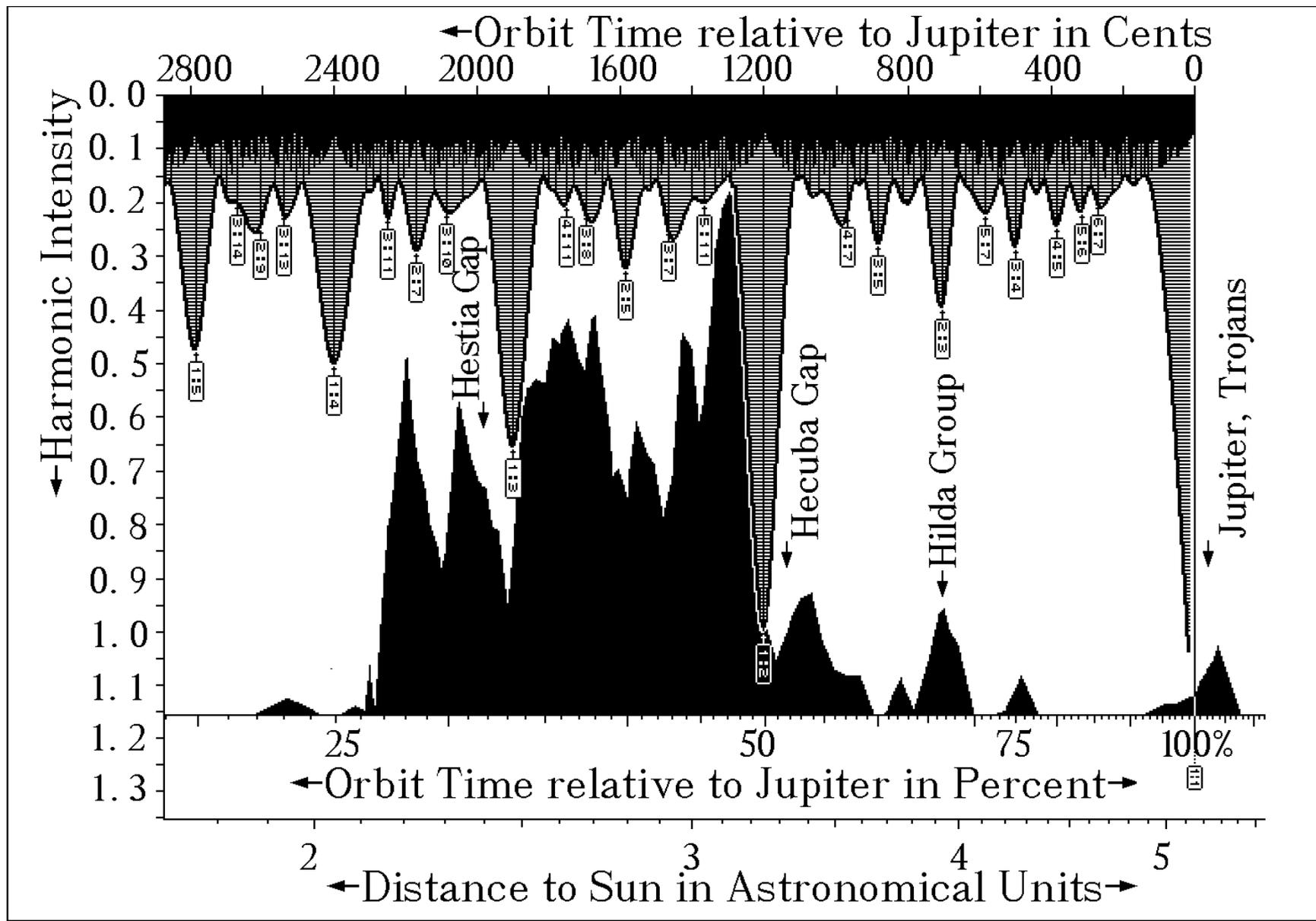
whereby $\text{sgn}(x) = -1$ for $x < 0$, else $\text{sgn}(x) = +1$

Interval-size (Ct)	Prime Decomposition as Powers of						Number-ratio	Harmonicity
	2	3	5	7	11	13		
0.000	0	0	0	0	0	0	1:1	$+\infty$
70.672	-3	-1	+2	0	0	0	24:25	+0.054152
111.731	+4	-1	-1	0	0	0	15:16	-0.076531
182.404	+1	-2	+1	0	0	0	9:10	+0.078534
203.910	-3	+2	0	0	0	0	8:9	+0.120000
231.174	+3	0	0	-1	0	0	7:8	-0.075269
266.871	-1	-1	0	+1	0	0	6:7	+0.071672
294.135	+5	-3	0	0	0	0	27:32	-0.076923
315.641	+1	+1	-1	0	0	0	5:6	-0.099338
386.314	-2	0	+1	0	0	0	4:5	+0.119048
407.820	-6	+4	0	0	0	0	64:81	+0.060000
427.373	+5	0	-2	0	0	0	25:32	-0.056180
435.084	0	+2	0	-1	0	0	7:9	-0.064024
470.781	-4	+1	0	+1	0	0	16:21	+0.058989
498.045	+2	-1	0	0	0	0	3:4	-0.214286
519.551	-2	+3	-1	0	0	0	20:27	-0.060976
568.717	-1	-2	+2	0	0	0	18:25	+0.052265
582.512	0	0	-1	+1	0	0	5:7	+0.059932
590.224	-5	+2	+1	0	0	0	32:45	+0.059761
609.776	+6	-2	-1	0	0	0	45:64	-0.056391
617.488	+1	0	+1	-1	0	0	7:10	-0.056543
680.449	+3	-3	+1	0	0	0	27:40	+0.057471
701.955	-1	+1	0	0	0	0	2:3	+0.272727
729.219	+5	-1	0	-1	0	0	21:32	-0.055703
764.916	+1	-2	0	+1	0	0	9:14	+0.060172
772.627	-4	0	+2	0	0	0	16:25	+0.059524
792.180	+7	-4	0	0	0	0	81:128	-0.056604
813.686	+3	0	-1	0	0	0	5:8	-0.106383
884.359	0	-1	+1	0	0	0	3:5	+0.110294
905.865	-4	+3	0	0	0	0	16:27	+0.083333
933.129	+2	+1	0	-1	0	0	7:12	-0.066879
968.826	-2	0	0	+1	0	0	4:7	+0.081395
996.090	+4	-2	0	0	0	0	9:16	-0.107143
1017.596	0	+2	-1	0	0	0	5:9	-0.085227
1088.269	-3	+1	+1	0	0	0	8:15	+0.082873
1129.328	+4	+1	-2	0	0	0	25:48	-0.051370
1137.039	-1	+3	0	-1	0	0	14:27	-0.051852
1200.000	+1	0	0	0	0	0	1:2	+1.000000

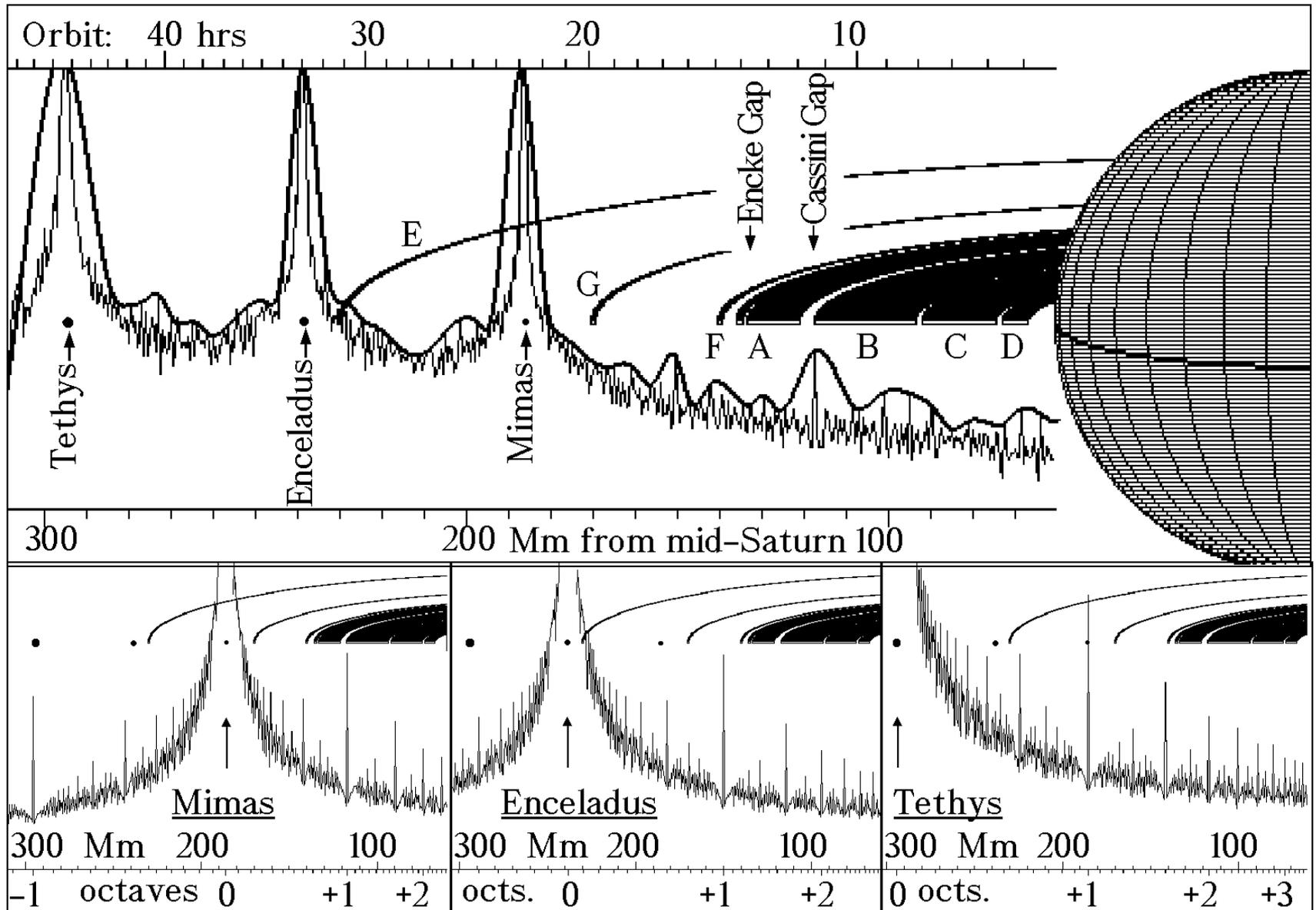
Graphical depiction of Harmonicity (three octaves)



Relation of Harmonicity to Asteroid Belt Density



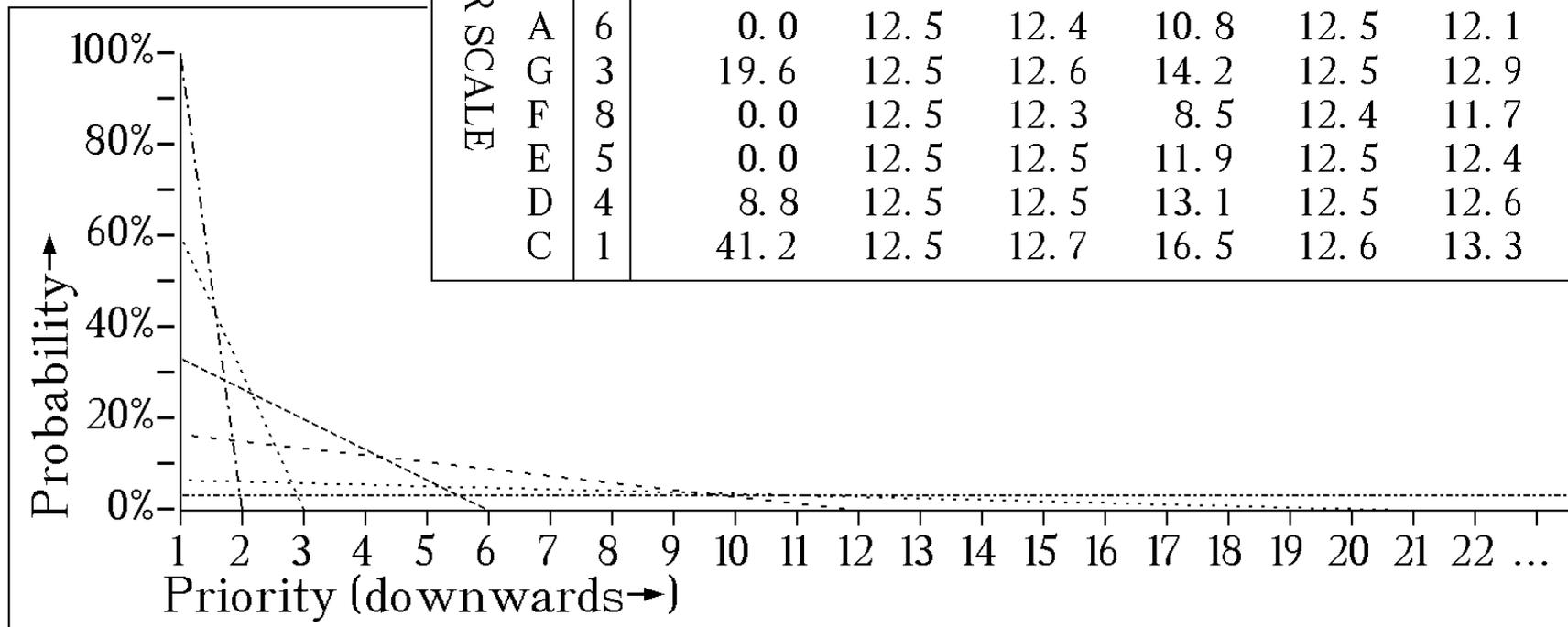
Relationship of Harmonicity to Saturn's rings



A Compositional Application of Harmonicity (e.g. in *Autobusk*)

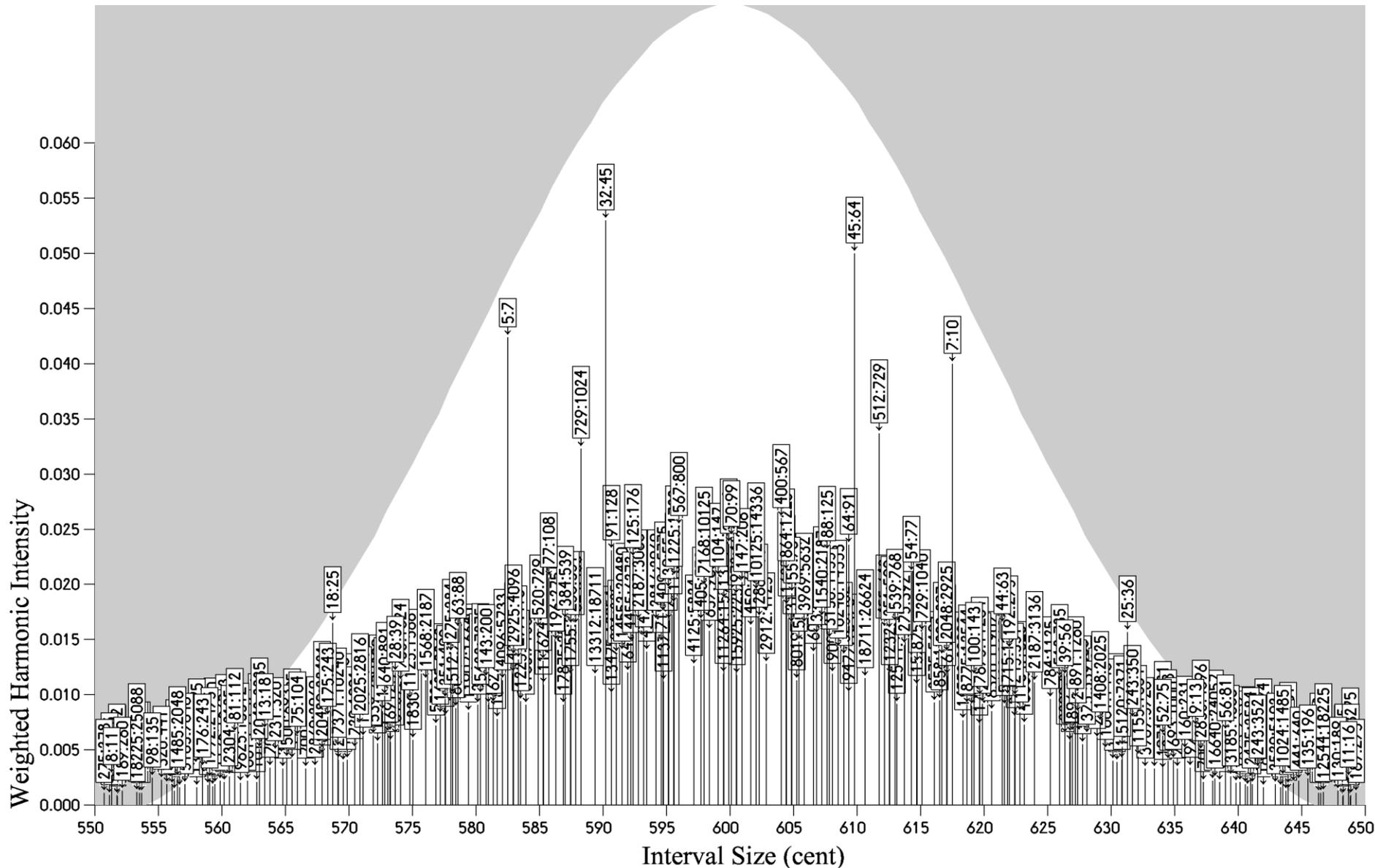
Pitch-Pulse matrix for a Major Scale and a $6/8$ Meter

Pitch:		SIX-EIGHT METER						
		Pulse:	1	2	3	4	5	6
		<i>Priority:</i>	1	6	4	2	5	3
MAJOR SCALE	C [♭]	2	30.4	12.5	12.6	15.3	12.5	13.1
	B	7	0.0	12.5	12.4	9.7	12.5	11.9
	A	6	0.0	12.5	12.4	10.8	12.5	12.1
	G	3	19.6	12.5	12.6	14.2	12.5	12.9
	F	8	0.0	12.5	12.3	8.5	12.4	11.7
	E	5	0.0	12.5	12.5	11.9	12.5	12.4
	D	4	8.8	12.5	12.5	13.1	12.5	12.6
C	1	41.2	12.5	12.7	16.5	12.6	13.3	

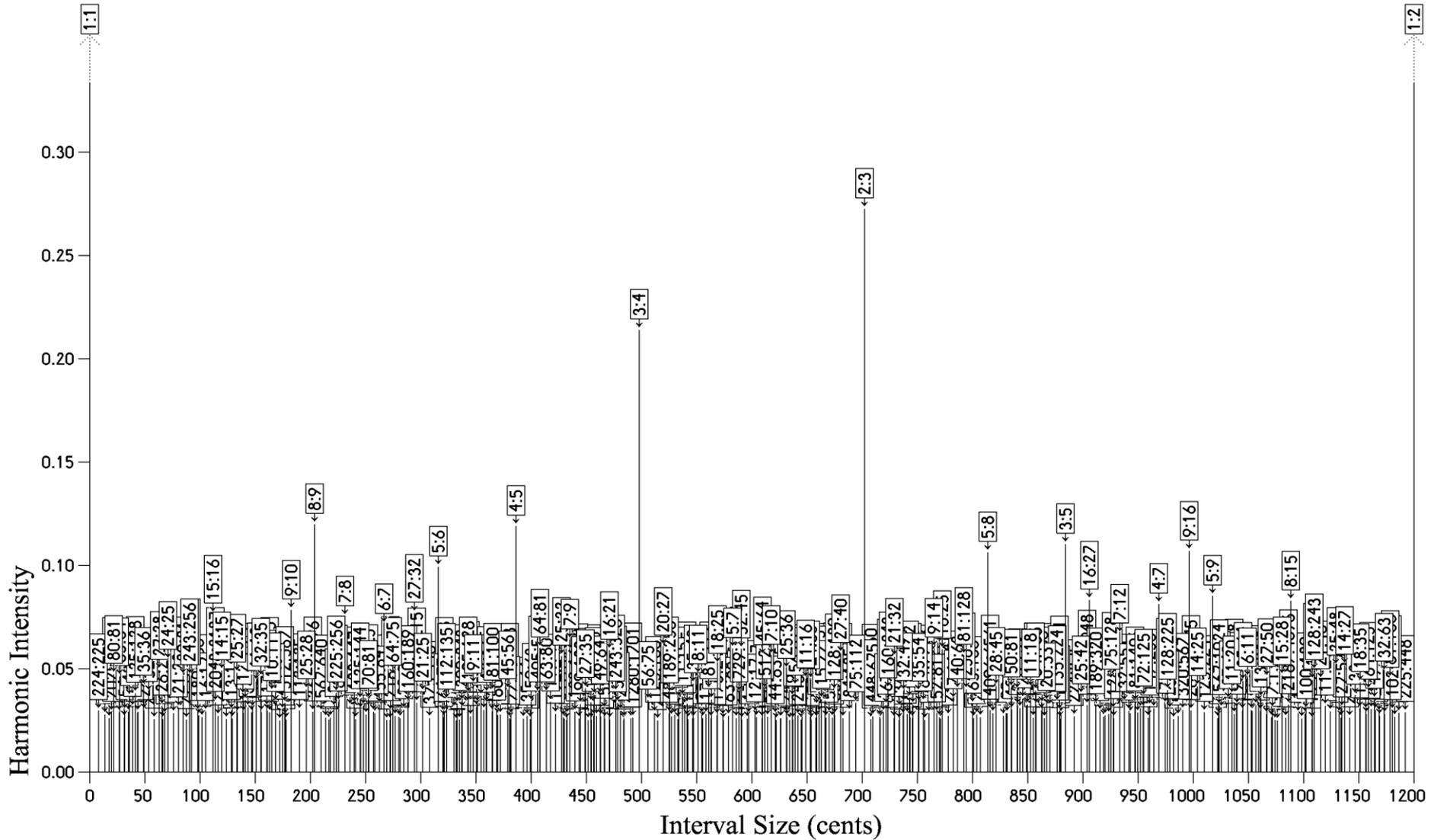


Harmonicities from 550-650 cents

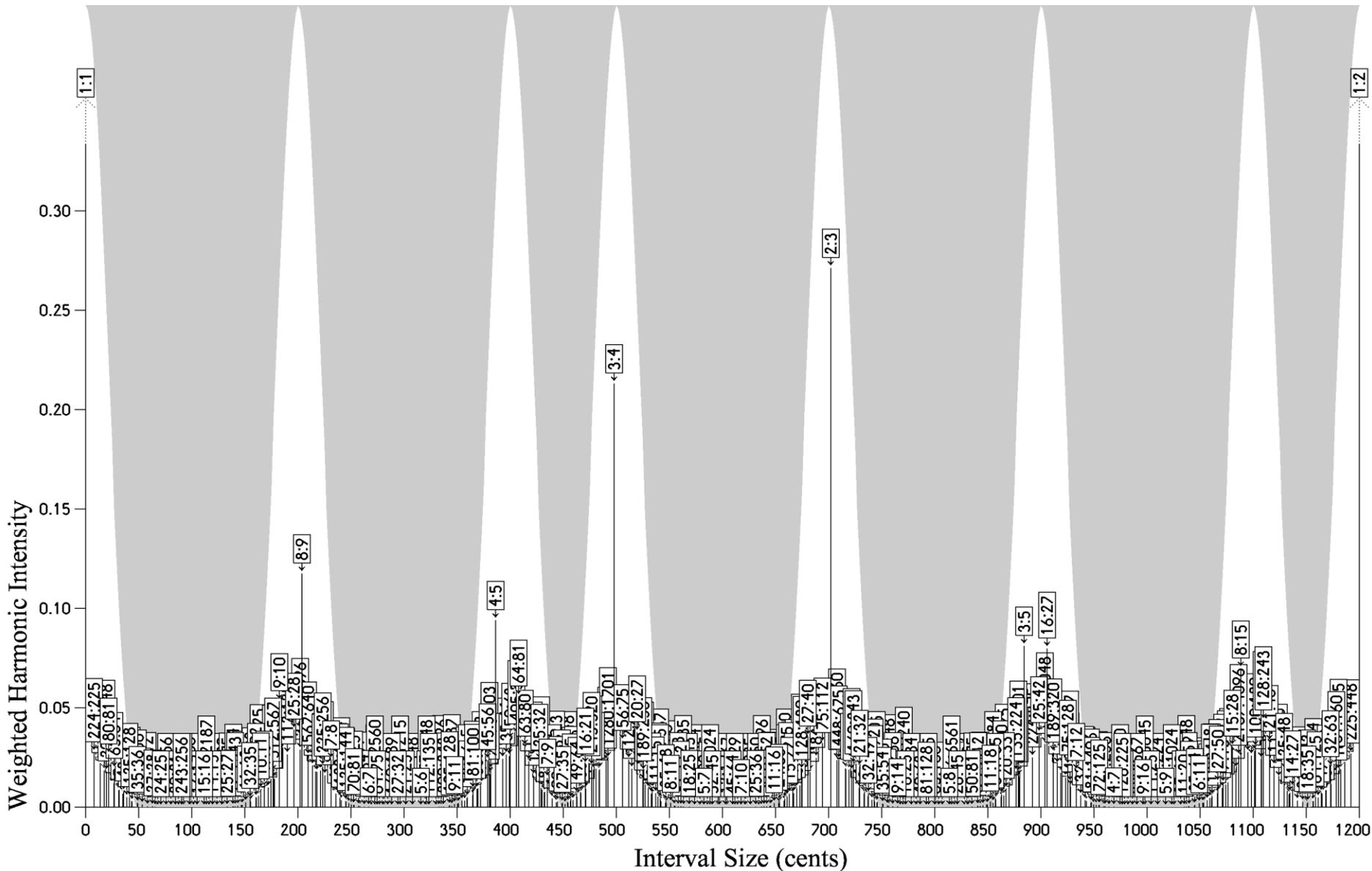
(weighted at the tritone)



Harmonicities (one octave, unweighted)



Harmonicities (one octave, weighted at the major scale)



Autobusk-Rationalization of the major scale: Specific Harmonicity

The tuning of one octave of a major scale finds the following to be the four most harmonic alternative tunings (AT) for each of the 8 pitches, using a minimum harmonicity (MH) of e.g. 0.04 and a nominal tolerance (NT) of e.g. 50 Ct:

1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1
80/81	10/9	81/64	27/20	40/27	27/16	243/128	81/40
81/80	8/7	56/45	21/16	243/160	128/75	256/135	160/81
63/64	35/32	32/25	48/35	32/21	12/7	40/21	63/32

ivl#	cents	dev.	ratio
1:	0	0	1/1
2:	200	4	9/8
3:	400	-14	5/4
4:	500	-2	4/3
5:	700	2	3/2
6:	900	-16	5/3
7:	1100	-12	15/8
8:	1200	0	2/1

Formula for Specific Harmonicity \mathcal{H}_s

$$\mathcal{H}_s = 1 / \min \left(\frac{\sum \xi(p)}{n(n-1)} \right)$$

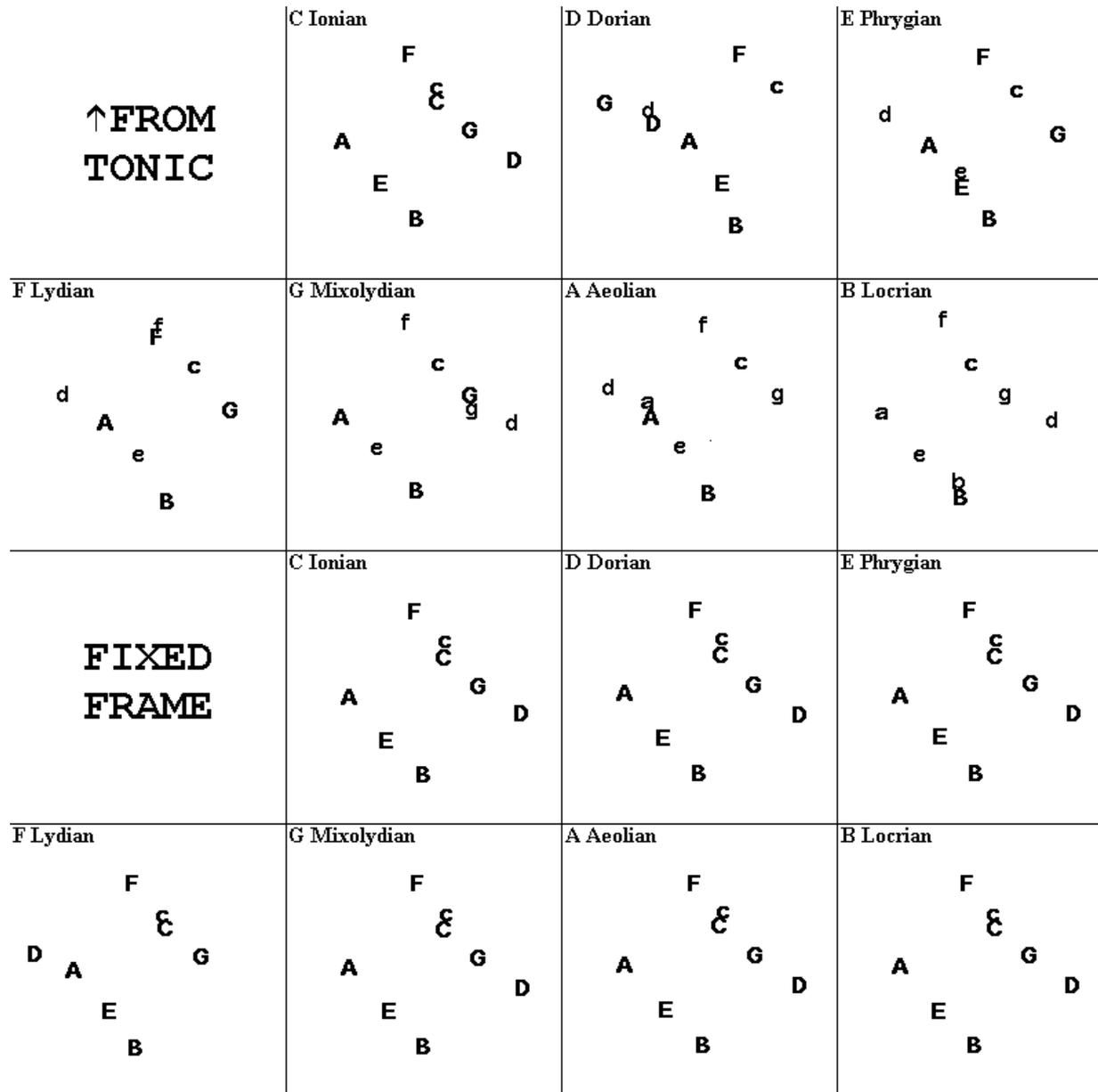
where n is the number of scale degrees, and ξ is the indigestibility of prime p

Matrix of harmonic intensity

	2~	3~	4~	5~	6~	7~	8~	
9/8		5/4	4/3	3/2	5/3	15/8	2/1	~1
0.120		0.119	-0.214	0.273	0.110	0.083	1.000	
		10/9	32/27	4/3	40/27	5/3	16/9	~2
		0.079	-0.077	-0.214	0.057	0.110	-0.107	
			16/15	6/5	4/3	3/2	8/5	~3
			-0.077	-0.099	-0.214	0.273	-0.106	
				9/8	5/4	45/32	3/2	~4
				0.120	0.119	0.060	0.273	
					10/9	5/4	4/3	~5
					0.079	0.119	-0.214	
						9/8	6/5	~6
						0.120	-0.099	
							16/15	~7
							-0.077	

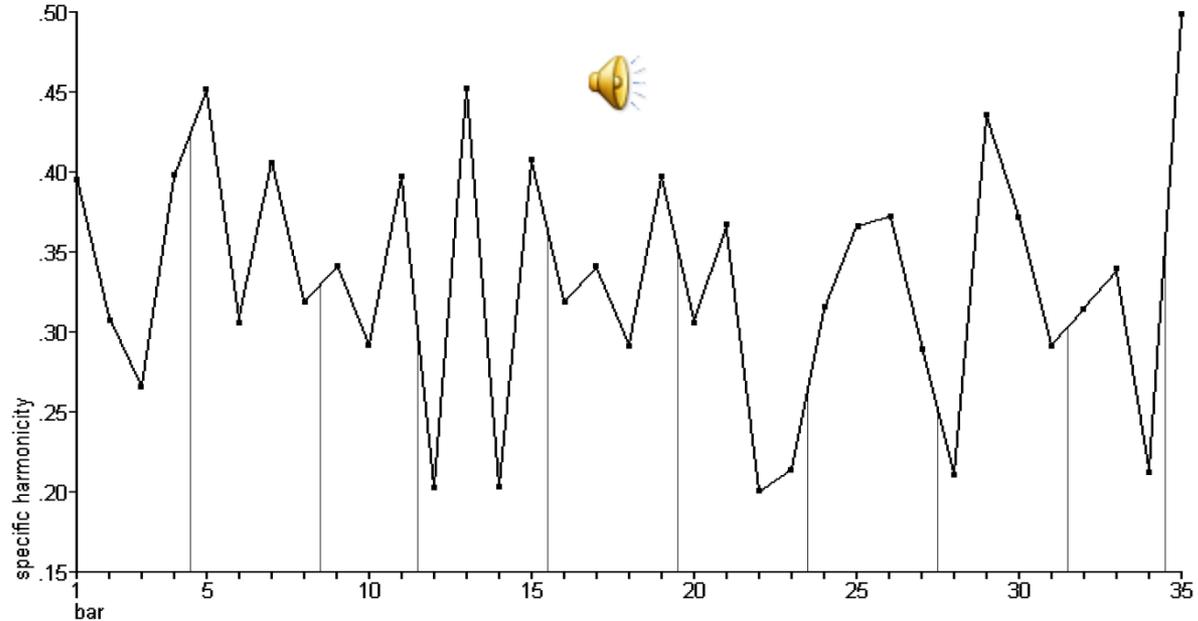
Specific Harmonicity: 0.2252

MDS of 7 church modes (movable or fixed frame)



Chord Tunings and Specific Harmonicity Fluctuations in *Prelude#1 of The Well-Tempered Clavier* by J.S.Bach

#	-2400	-1900	-1800	-1700	-1600	-1200	-1100	-1000	-900	-800	-700	-500	-400	-300	-200	-100	0	200	300	400	500	600	700	900	1100	1200	1300	1400	1600	1700	1900	2100	sp.	harm.		
1	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	1/1	---	---	5/4	---	---	3/2	---	---	2/1	---	---	---	---	---	---	---	---	0.3968	
2	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	1/1	10/9	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3088	
3	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	15/16	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.2667	
4	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	1/1	---	---	5/4	---	---	3/2	---	---	2/1	---	---	---	---	---	---	---	---	0.3968	
5	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	1/1	---	---	5/4	---	---	3/2	---	---	2/1	---	---	---	---	---	---	---	---	0.4505	
6	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	1/1	9/8	---	---	---	---	45/32	---	27/16	---	---	---	---	9/4	---	---	---	0.3067		
7	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	15/16	---	---	---	---	---	---	3/2	---	---	---	---	---	---	---	---	---	---	---	0.4065	
8	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	15/16	1/1	---	---	---	---	---	3/2	---	---	---	---	---	---	---	---	---	10/3	---	0.4505	
9	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	1/1	---	---	5/4	---	---	3/2	---	---	2/1	---	---	---	---	---	---	---	---	0.3396	
10	---	---	---	---	---	---	---	9/16	---	---	---	---	---	---	---	27/32	---	---	---	---	---	---	45/32	---	---	2/1	---	---	---	---	---	---	---	---	0.2923	
11	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	15/16	---	---	---	---	---	3/2	---	---	15/8	---	---	---	---	---	---	---	---	0.3968	
12	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	9/10	---	---	---	---	---	---	---	---	---	---	---	135/64	---	---	---	---	---	---	0.2029	
13	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	27/32	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.4505	
14	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	4/5	---	---	---	---	---	---	4/3	---	---	15/8	---	---	---	---	---	---	---	---	0.2029	
15	---	---	---	---	---	---	---	---	---	5/8	---	3/4	---	---	---	---	1/1	---	---	---	---	---	3/2	---	---	2/1	---	---	---	---	---	---	---	---	0.4065	
16	---	---	---	---	---	---	---	---	---	5/8	---	3/4	---	---	---	---	1/1	---	---	---	---	---	4/3	---	---	---	---	---	---	---	---	---	---	---	0.3193	
17	---	---	---	---	---	---	---	---	---	2/3	---	---	---	---	---	---	5/6	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3396	
18	---	---	---	---	---	---	---	---	---	5/8	---	3/4	---	---	---	---	15/16	---	---	---	---	---	4/3	---	---	---	---	---	---	---	---	---	---	---	---	0.2923
19	---	---	---	3/8	---	---	---	9/16	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3968
20	---	---	---	---	---	---	---	---	---	5/8	---	3/4	---	---	---	---	8/9	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3067
21	---	---	1/3	---	---	---	---	---	---	---	---	---	---	---	---	---	5/6	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3660	
22	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.1996
23	---	---	16/45	---	---	---	---	---	---	---	---	---	---	---	---	---	5/6	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.2147
24	---	---	---	3/8	---	---	---	---	---	---	---	---	---	---	---	---	15/16	1/1	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3144
25	---	---	---	3/8	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3660
26	---	---	---	3/8	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3719
27	---	---	---	3/8	---	---	---	9/16	---	---	---	---	---	---	---	---	15/16	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.2923
28	---	---	---	3/8	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.2111
29	---	---	---	3/8	---	---	---	---	---	5/8	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.4348
30	---	---	---	3/8	---	---	---	9/16	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3719
31	---	---	---	3/8	---	---	---	9/16	---	---	---	---	---	---	---	---	15/16	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.2923
32	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3144
33	---	---	1/4	---	---	---	---	1/2	---	---	---	---	---	---	---	---	5/6	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.3383
34	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.2141
35	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	0.4980



Chord Structure in Prelude #1 of The Well-Tempered Clavier by J.S.Bach

Half-matrix of intra-scalar intervals in rationalized 12- and 13-Tone equal-tempered scales

Rationalized Tuning Network of Equal-tempered Scales with Intrascalar Ratios and Harmonicities

Scale Degree:	1	2	3	4	5	6	7	8	9	10	11	12	13
Size given (ct):	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
Tuning:	1/1	16/15	9/8	6/5	5/4	4/3	45/32	3/2	8/5	5/3	9/5	15/8	2/1
Deviation (ct):	0	+12	+4	+16	-14	-2	-10	+2	+14	-16	+18	-12	0
Note name:	c	cb ¹	d ²	eb ¹	e	f ₁	f ²	g ¹	ab ¹	a ₁	bb ²	b ¹	c

Specific Harmonicity: 0.1614

Twelve Tone Matrix:													
2→	3→	4→	5→	6→	7→	8→	9→	10→	11→	12→	13→		
16/15	9/8	6/5	5/4	4/3	45/32	3/2	8/5	5/3	9/5	15/8	2/1	→1	
-0.077	+0.120	-0.099	+0.119	-0.214	+0.060	+0.273	-0.106	+0.110	-0.085	+0.083	+1.000		
	135/128	9/8	75/64	5/4	675/512	45/32	3/2	25/16	27/16	225/128	15/8	→2	
	+0.047	+0.120	+0.047	+0.119	+0.034	+0.060	+0.273	+0.060	+0.083	+0.040	+0.083		
		16/15	10/9	32/27	5/4	4/3	64/45	40/27	8/5	5/3	16/9	→3	
Thirteen Tone Matrix:		-0.077	+0.079	-0.077	+0.119	-0.214	-0.056	+0.057	-0.106	+0.110	-0.107		
	256/243		+0.054	+0.079	+0.047	+0.119	-0.214	+0.052	+0.273	+0.060	+0.110	→4	
13→	-0.047			16/15	9/8	6/5	32/25	4/3	36/25	3/2	8/5	→5	
12→	10/9	135/128		-0.077	+0.120	-0.099	-0.056	-0.214	-0.050	+0.273	-0.106		
	+0.079	+0.047			135/128	9/8	6/5	5/4	27/20	45/32	3/2	→6	
11→	32/27	9/8	16/15		+0.047	+0.120	-0.099	+0.119	-0.061	+0.060	+0.273		
	-0.077	+0.120	-0.077			16/15	256/225	32/27	32/25	4/3	64/45	→7	
10→	5/4	1215/1024	9/8	135/128		-0.077	-0.038	-0.077	-0.056	-0.214	-0.056		
	+0.119	+0.034	+0.120	+0.047			16/15	10/9	6/5	5/4	4/3	→8	
9→	21/16	5103/4096	189/160	567/512	21/20		-0.077	+0.079	-0.099	+0.119	-0.214		
	+0.059	+0.026	+0.034	+0.033	+0.047			25/24	9/8	75/64	5/4	→9	
8→	48/35	729/560	216/175	81/70	192/175	256/245		+0.054	+0.120	+0.047	+0.119		
	-0.043	-0.027	-0.029	-0.035	-0.031	-0.029			27/25	9/8	6/5	→10	
7→	35/24	2835/2048	21/16	315/256	7/6	10/9	1225/1152		-0.048	+0.120	-0.099		
	+0.045	+0.026	+0.059	+0.033	+0.072	+0.079	+0.022			25/24	10/9	→11	
6→	32/21	81/56	48/35	9/7	128/105	512/441	10/9	256/245		+0.054	+0.079		
	-0.056	-0.042	-0.043	-0.064	-0.038	-0.029	+0.079	-0.029			16/15	→12	
5→	8/5	243/160	36/25	27/20	32/25	128/105	7/6	192/175	21/20		-0.077		
	-0.106	+0.040	-0.050	-0.061	-0.056	-0.038	+0.072	-0.031	+0.047				
4→	12/7	729/448	54/35	81/56	48/35	64/49	5/4	288/245	9/8	15/14			
	-0.067	-0.031	-0.039	-0.042	-0.043	-0.038	+0.119	-0.027	+0.120	-0.049			
3→	16/9	27/16	8/5	3/2	64/45	256/189	35/27	128/105	7/6	10/9	28/27		
	-0.107	+0.083	-0.106	+0.273	-0.056	-0.038	+0.041	-0.038	+0.072	+0.079	+0.049		
2→	256/135	9/5	128/75	8/5	1024/675	4096/2835	112/81	2048/1575	56/45	32/27	448/405	16/15	
	-0.045	-0.085	-0.045	-0.106	-0.032	-0.025	+0.040	-0.025	+0.040	-0.077	-0.030	-0.077	
1→	2/1	243/128	9/5	27/16	8/5	32/21	35/24	48/35	21/16	5/4	7/6	9/8	135/128
	+1.000	+0.049	-0.085	+0.083	-0.106	-0.056	+0.045	-0.043	+0.059	+0.119	+0.072	+0.120	+0.047
	→14	→13	→12	→11	→10	→9	→8	→7	→6	→5	→4	→3	→2

Scale Degree:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Size given (Ct):	0	92	185	277	369	462	554	646	738	831	923	1015	1108	1200
Tuning:	1/1	135/128	9/8	7/6	5/4	21/16	48/35	35/24	32/21	8/5	27/16	9/5	243/128	2/1
Deviation (Ct):	0	0	+19	-10	+17	+9	-7	+7	-9	-17	-17	+3	+2	0
Note name:	c	c ²	d ²	eb ²	e	f ²	g ²	ab ²	a ₁	bb ²	a ³	bb ²	b ⁵	c

Specific Harmonicity: 0.0956

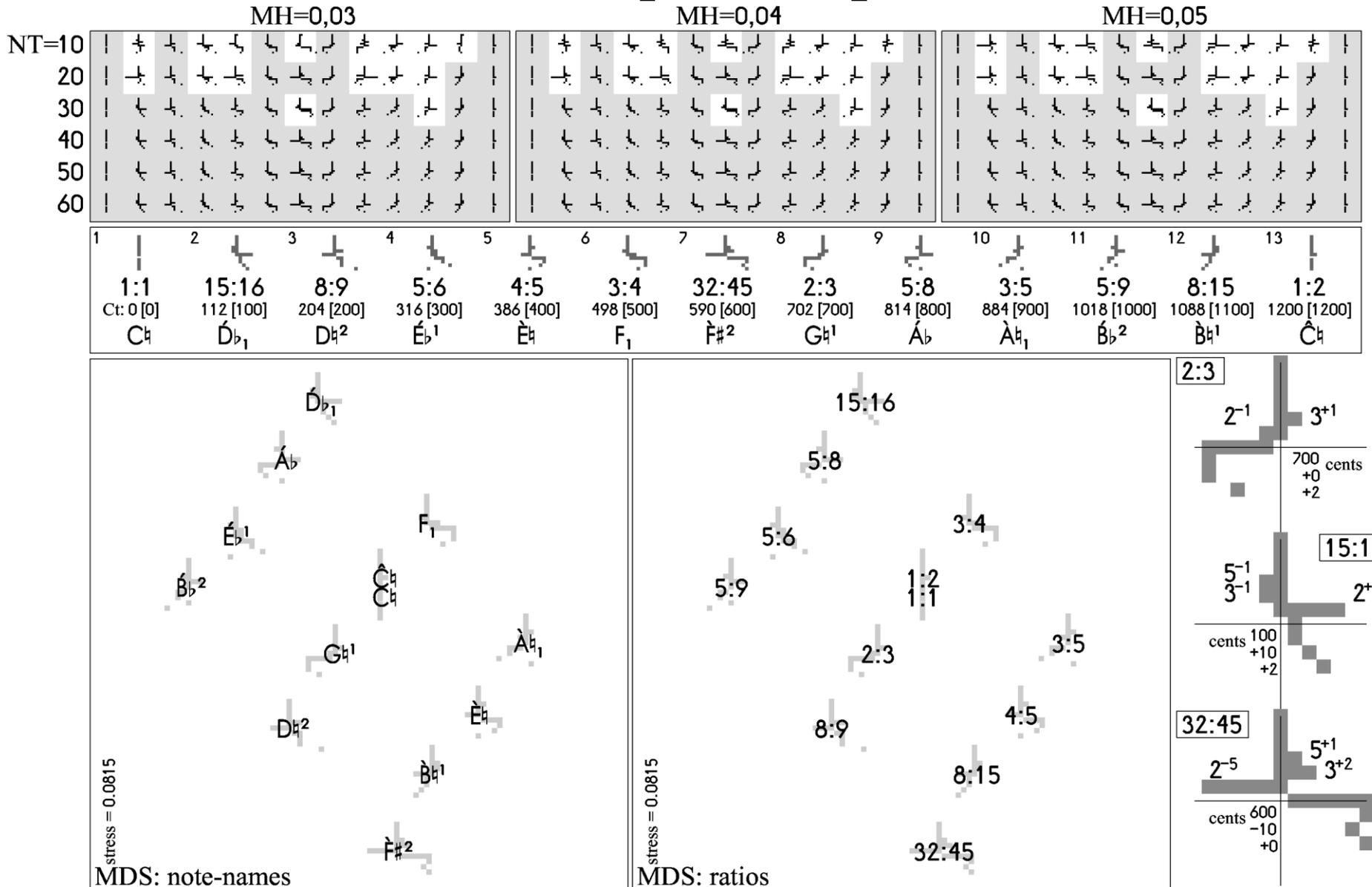
Four sets of alternative tunings for the 13 scale-degrees of a 12-tone equal tempered scale, with selections

```

6 primes (^ max.abs.powers)
  2 ^ 11
  3 ^ 7
  5 ^ 3
  7 ^ 2
  11 ^ 1
  13 ^ 1
30000 = numer\denom limit
  100 = indigestibility limit
  9999 cents range
  18113 intervals' choice
~~~~~
4 tunings per pitch
50 cents tolerance
Alternatives:
  0: →    1/1          224/225          225/224          80/81
 100: →   16/15       135/128          256/243          21/20
 200: →    9/8         10/9             28/25            125/112
 300:    32/27   →    6/5             25/21            1215/1024
 400: →    5/4         81/64           512/405          63/50
 500: →    4/3         27/20            75/56            1701/1280
 600: →   45/32        64/45             7/5              10/7
 700: →    3/2         40/27           112/75           125/84
 800: →    8/5         128/81           405/256          100/63
 900: →    5/3         27/16            42/25            2048/1215
1000:    16/9   →    9/5             25/14            3645/2048
1100: →   15/8        243/128          256/135           40/21
1200: →    2/1         225/112          81/40             448/225
Best constellation : 1112111111211
0.1614 specific harmonicity

```

Multi-dimensional scaling of a rationalization of the 12-tone equal-tempered scale



Four sets of alternative tunings for the 14 scale-degrees of a 13-tone equal tempered scale, with selections

6 primes ([^] max.abs.powers) for two and four alternative tunings

```

2 ^ 11
3 ^ 7
5 ^ 3
7 ^ 2
11 ^ 1
13 ^ 1
30000 = numer\denom limit
100 = indigestibility limit
9999 cents range
18113 intervals' choice

```

~~~~~

2 tunings per pitch  
40 cents tolerance  
Alternatives:

|       |   |         |   |         |
|-------|---|---------|---|---------|
| 0:    | → | 1/1     |   | 224/225 |
| 92:   | → | 135/128 |   | 256/243 |
| 185:  |   | 10/9    | → | 9/8     |
| 277:  | → | 7/6     |   | 75/64   |
| 369:  | → | 5/4     |   | 100/81  |
| 462:  | → | 21/16   |   | 64/49   |
| 554:  |   | 11/8    | → | 48/35   |
| 646:  |   | 16/11   | → | 35/24   |
| 738:  | → | 32/21   |   | 49/32   |
| 831:  | → | 8/5     |   | 81/50   |
| 923:  |   | 12/7    | → | 27/16   |
| 1015: | → | 9/5     |   | 16/9    |
| 1108: | → | 243/128 |   | 256/135 |
| 1200: | → | 2/1     |   | 225/112 |

Best constellation : 11211122112111  
0.0956 specific harmonicity

4 tunings per pitch  
40 cents tolerance  
Alternatives:

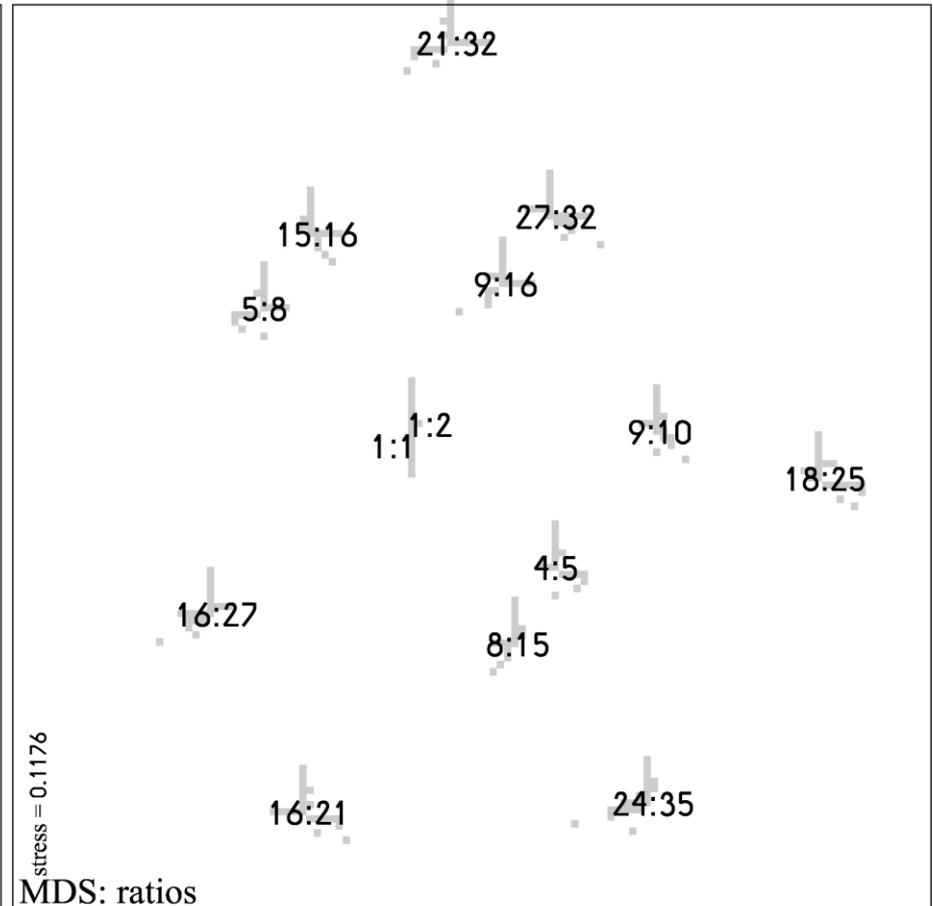
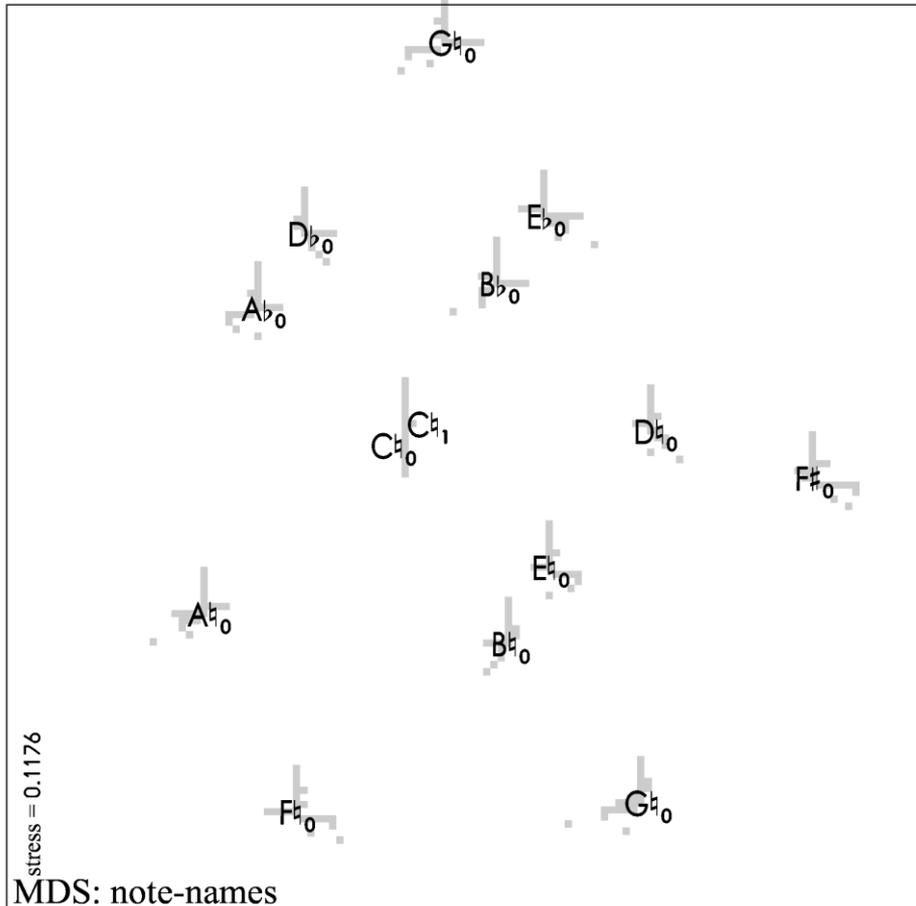
|       |   |         |   |         |   |         |   |           |
|-------|---|---------|---|---------|---|---------|---|-----------|
| 0:    | → | 1/1     |   | 224/225 |   | 225/224 |   | 5103/5120 |
| 92:   |   | 135/128 |   | 256/243 |   | 21/20   | → | 16/15     |
| 185:  | → | 10/9    |   | 9/8     |   | 28/25   |   | 567/512   |
| 277:  |   | 7/6     |   | 75/64   | → | 32/27   |   | 2560/2187 |
| 369:  | → | 5/4     |   | 100/81  |   | 56/45   |   | 16/13     |
| 462:  | → | 21/16   |   | 64/49   |   | 125/96  |   | 13/10     |
| 554:  |   | 11/8    |   | 48/35   |   | 112/81  | → | 25/18     |
| 646:  |   | 16/11   | → | 35/24   |   | 81/56   |   | 36/25     |
| 738:  | → | 32/21   |   | 49/32   |   | 192/125 |   | 55/36     |
| 831:  | → | 8/5     |   | 81/50   |   | 45/28   |   | 13/8      |
| 923:  |   | 12/7    | → | 27/16   |   | 128/75  |   | 2187/1280 |
| 1015: | → | 9/5     | → | 16/9    |   | 25/14   |   | 1024/567  |
| 1108: |   | 243/128 |   | 256/135 |   | 40/21   | → | 15/8      |
| 1200: | → | 2/1     |   | 225/112 |   | 448/225 |   | 5103/2560 |

Best constellation : 14131142112241  
0.1074 specific harmonicity

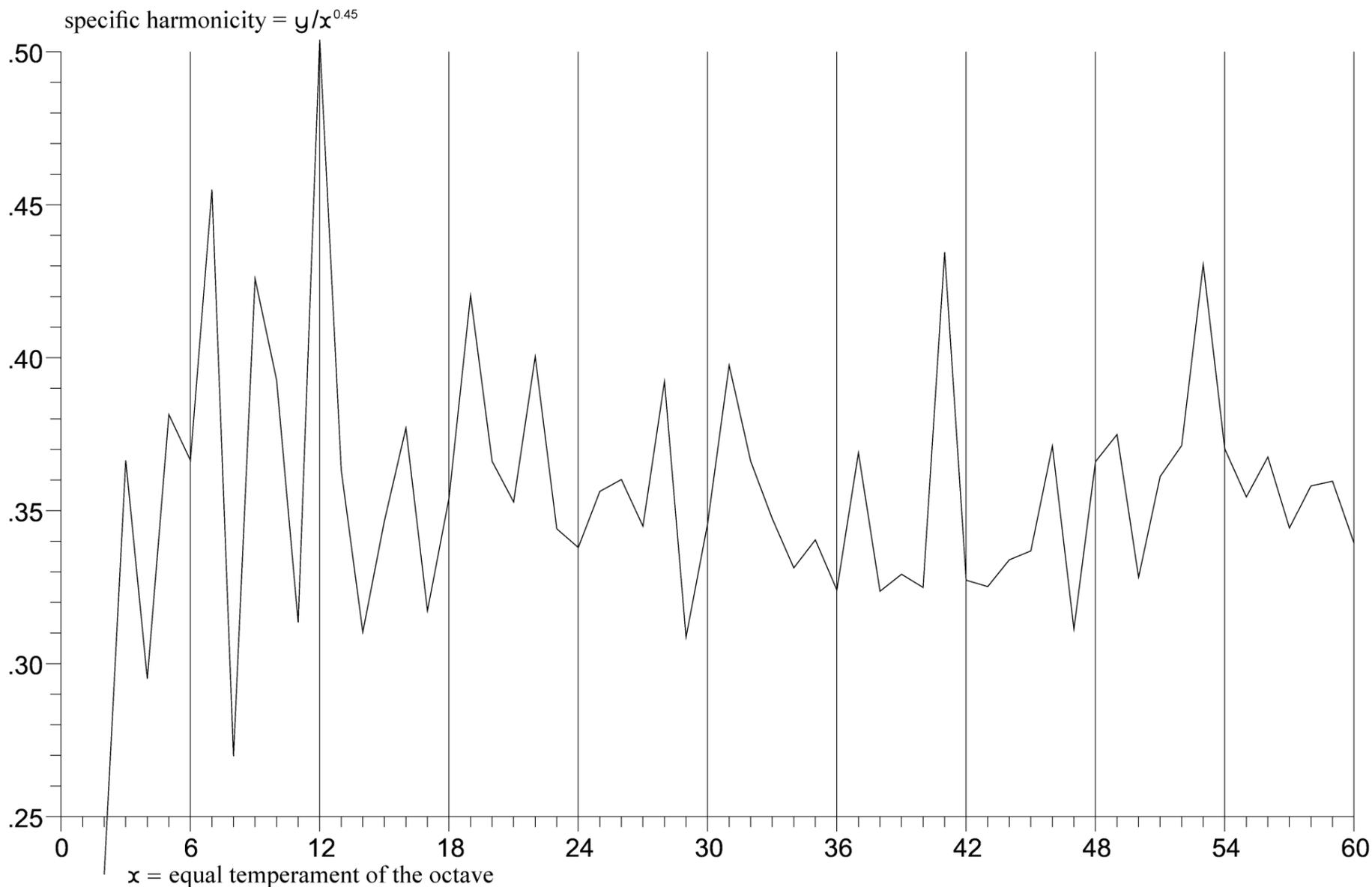


# Multi-dimensional scaling of a rationalization of the 13-tone equal-tempered scale (AT=4)

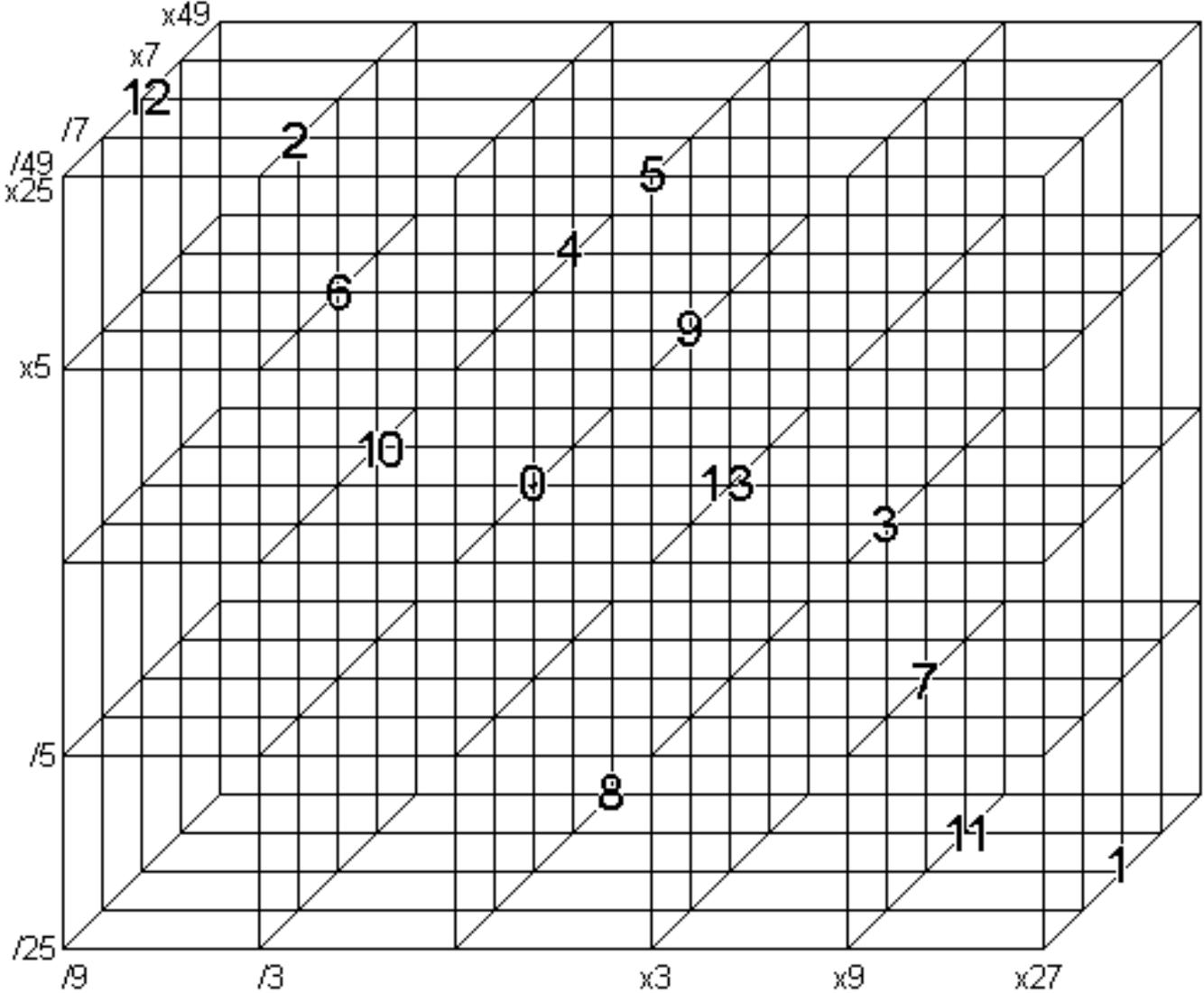
|                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |      |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|------|
| 1                           | 2                           | 3                           | 4                           | 5                           | 6                           | 7                           | 8                           | 9                           | 10                          | 11                          | 12                          | 13                          | 14                          |      |
| 1:1                         | 15:16                       | 9:10                        | 27:32                       | 4:5                         | 16:21                       | 18:25                       | 24:35                       | 21:32                       | 5:8                         | 16:27                       | 9:16                        | 8:15                        | 1:2                         |      |
| cents                       | 0                           | 112                         | 182                         | 294                         | 386                         | 471                         | 569                         | 653                         | 729                         | 814                         | 906                         | 996                         | 1088                        | 1200 |
| [cents]                     | [92]                        | [185]                       | [277]                       | [369]                       | [462]                       | [554]                       | [646]                       | [738]                       | [831]                       | [923]                       | [1015]                      | [1108]                      | [1200]                      |      |
| 0                           | +20                         | -3                          | +17                         | +17                         | +9                          | +15                         | +7                          | -9                          | -17                         | -17                         | -19                         | -20                         | 0                           |      |
| C <sub>♭</sub> <sub>0</sub> | D <sub>♭</sub> <sub>0</sub> | D <sub>♮</sub> <sub>0</sub> | E <sub>♭</sub> <sub>0</sub> | E <sub>♮</sub> <sub>0</sub> | F <sub>♭</sub> <sub>0</sub> | F <sub>♮</sub> <sub>0</sub> | G <sub>♭</sub> <sub>0</sub> | G <sub>♮</sub> <sub>0</sub> | A <sub>♭</sub> <sub>0</sub> | A <sub>♮</sub> <sub>0</sub> | B <sub>♭</sub> <sub>0</sub> | B <sub>♮</sub> <sub>0</sub> | C <sub>♮</sub> <sub>1</sub> |      |



# Specific Harmonicities of rationalized 2- to 60-tone equal-tempered scales



# 3-5-7 tuning lattice for the Bohlen-Pierce Scale



# Various *Autobusk*-rationalizations of the BP-Scale

Bohlen-Pierce-Scale (AT=2; Values for NT=40 and NT=50 identical):

MH=0.03

| NT | SH     | Tuning |       |       |       |       |       |     |          |       |       |       |          |        |     |
|----|--------|--------|-------|-------|-------|-------|-------|-----|----------|-------|-------|-------|----------|--------|-----|
| 10 | 0.0796 | 1/1    | 49/45 | 32/27 | 9/7   | 7/5   | 32/21 | 5/3 | 1024/567 | 63/32 | 32/15 | 7/3   | 1024/405 | 224/81 | 3/1 |
| 20 | 0.0861 | 1/1    | 12/11 | 32/27 | 9/7   | 7/5   | 32/21 | 5/3 | 9/5      | 63/32 | 32/15 | 7/3   | 81/32    | 224/81 | 3/1 |
| 30 | 0.0959 | 1/1    | 35/32 | 32/27 | 32/25 | 7/5   | 32/21 | 5/3 | 9/5      | 2/1   | 32/15 | 7/3   | 81/32    | 224/81 | 3/1 |
| 40 | 0.1200 | 1/1    | 35/32 | 6/5   | 9/7   | 45/32 | 3/2   | 5/3 | 9/5      | 2/1   | 15/7  | 75/32 | 5/2      | 25/9   | 3/1 |

MH=0.04

| NT | SH     | Tuning |       |       |       |       |         |     |     |       |       |       |       |       |     |
|----|--------|--------|-------|-------|-------|-------|---------|-----|-----|-------|-------|-------|-------|-------|-----|
| 10 | 0.0905 | 1/1    | 35/32 | 32/27 | 9/7   | 45/32 | 32/21   | 5/3 | 9/5 | 63/32 | 32/15 | 7/3   | 81/32 | 96/35 | 3/1 |
| 20 | 0.0912 | 1/1    | 35/32 | 75/64 | 9/7   | 45/32 | 243/160 | 5/3 | 9/5 | 63/32 | 15/7  | 75/32 | 81/32 | 96/35 | 3/1 |
| 30 | 0.1058 | 1/1    | 35/32 | 6/5   | 32/25 | 7/5   | 32/21   | 5/3 | 9/5 | 2/1   | 32/15 | 7/3   | 5/2   | 96/35 | 3/1 |
| 40 | 0.1200 | 1/1    | 35/32 | 6/5   | 9/7   | 45/32 | 3/2     | 5/3 | 9/5 | 2/1   | 15/7  | 75/32 | 5/2   | 25/9  | 3/1 |

[3-5-7]-Grid of other theorists:

1/1 27/25 25/21 9/7 7/5 75/49 5/3 9/5 49/25 15/7 7/3 63/25 25/9 3/1

Autobusk rationalization of BP scale  
Tolerance = 30 cents  
Four best candidates for

-----  
732 cents:

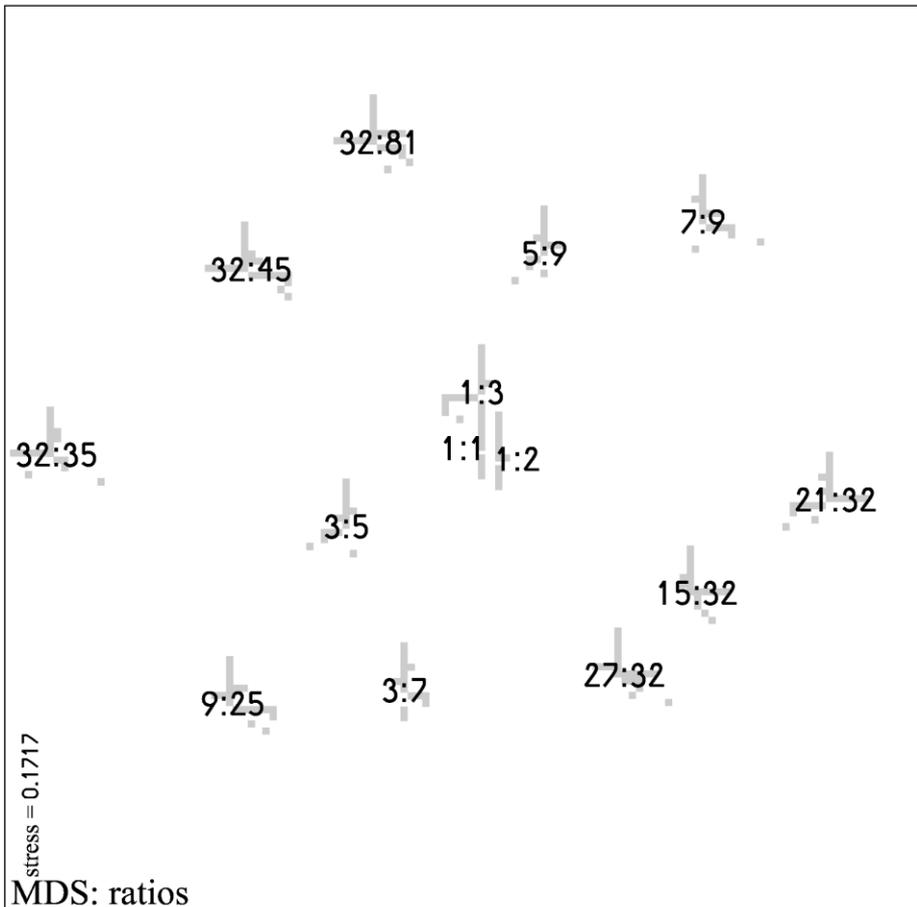
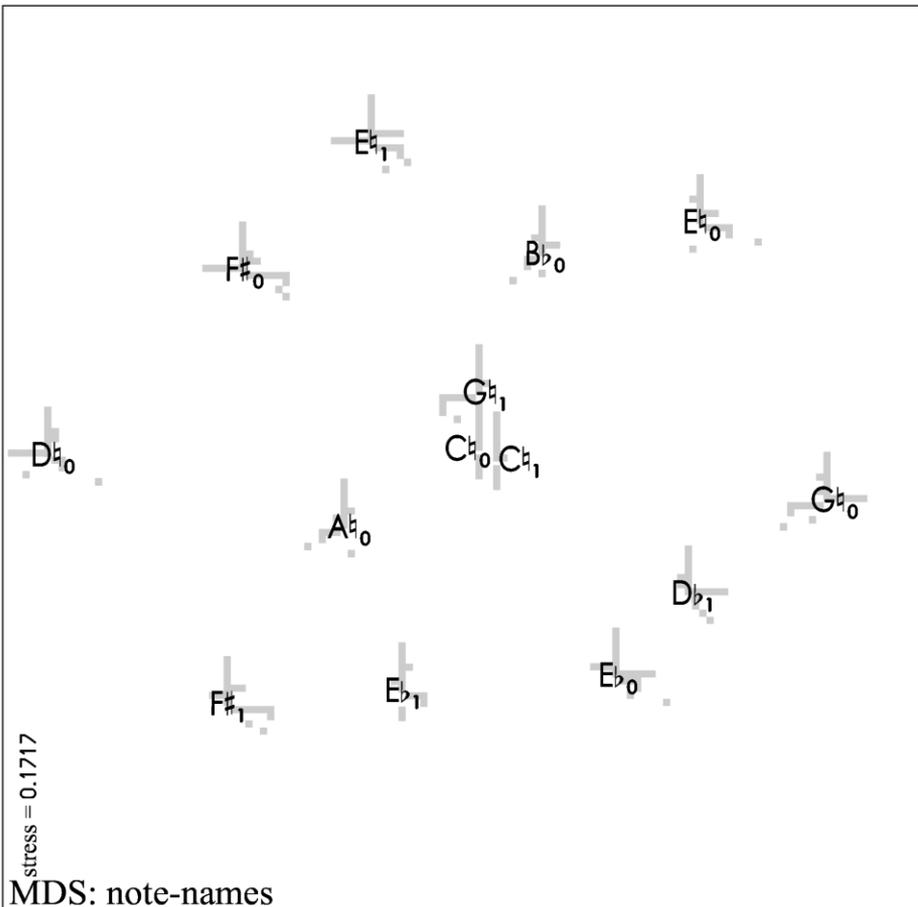
|                       | ratio   | size (cents) | harmonicity |
|-----------------------|---------|--------------|-------------|
| 1.                    | 32/21   | 729.2        | -0.055703   |
| 2.                    | 49/32   | 737.7        | +0.039106   |
| 3.                    | 243/160 | 723.5        | +0.040431   |
| 4.                    | 55/36   | 733.7        | +0.031333   |
| 3-5-7 lattice tuning: |         |              |             |
|                       | 75/49   | 736.9        | -0.027748   |

-----  
1170 cents:

|                       | ratio  | size (cents) | harmonicity |
|-----------------------|--------|--------------|-------------|
| 1.                    | 2/1    | 1200.0       | +1.000000   |
| 2.                    | 63/32  | 1172.7       | +0.048499   |
| 3.                    | 160/81 | 1178.5       | +0.045317   |
| 4.                    | 96/49  | 1164.3       | -0.035413   |
| 3-5-7 lattice tuning: |        |              |             |
|                       | 49/25  | 1165.0       | +0.029966   |

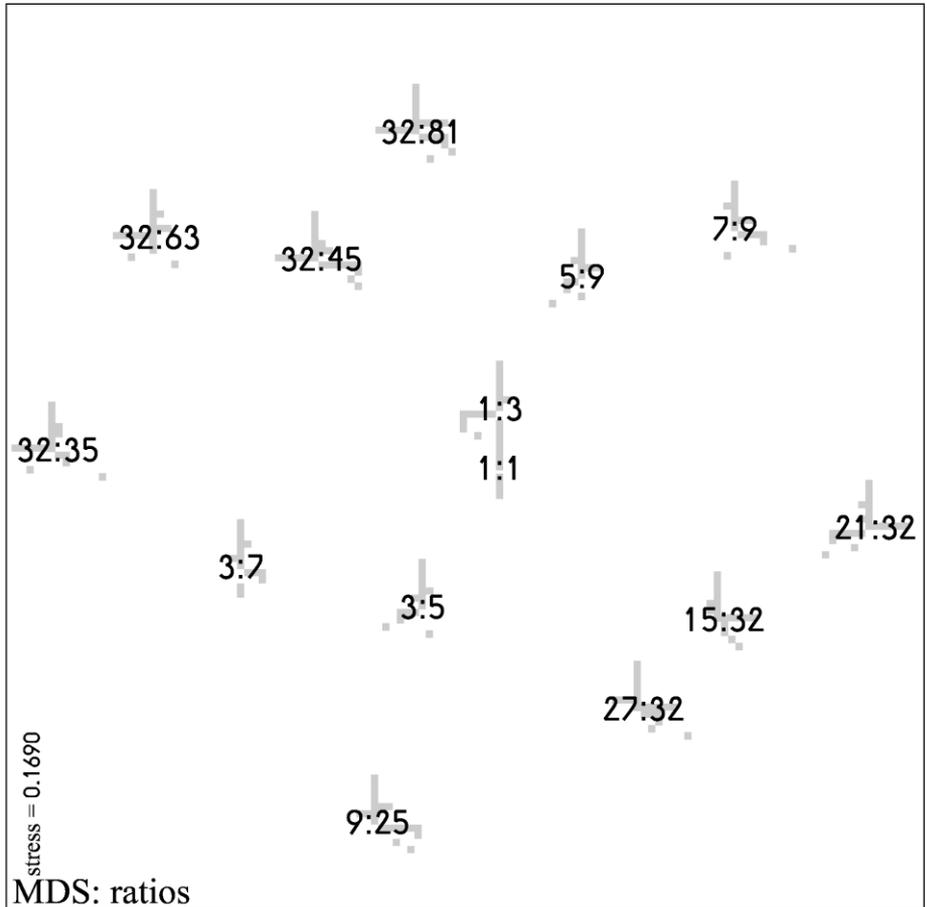
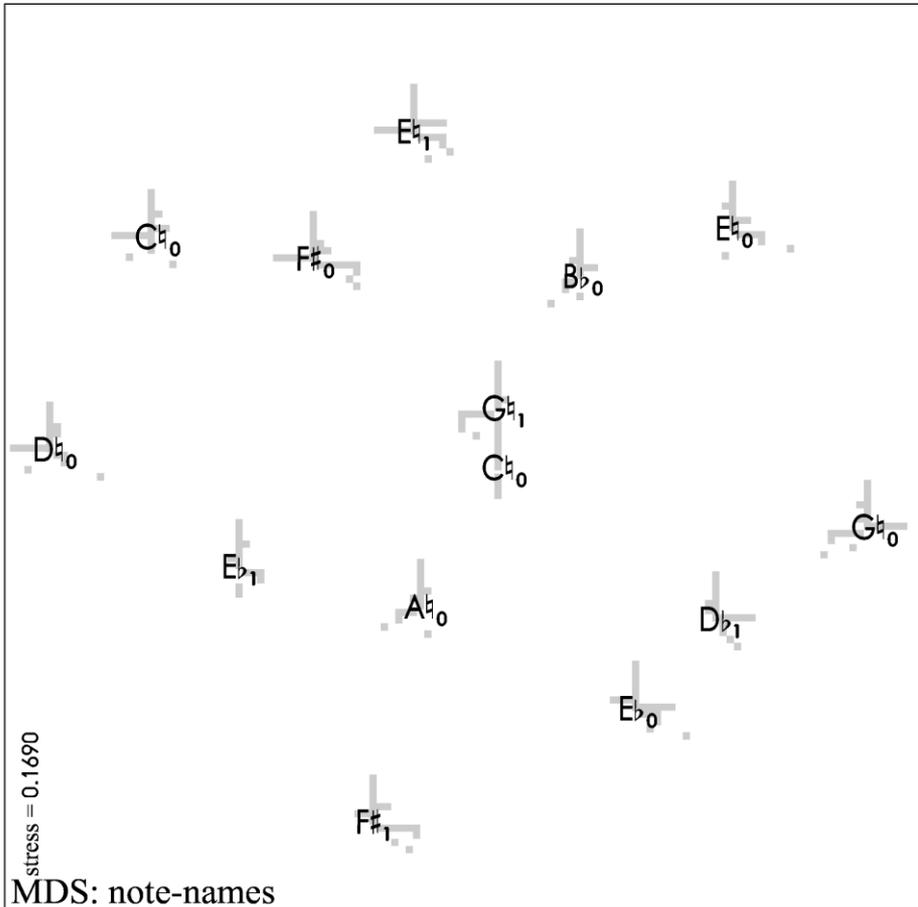
# MDS of a rationalization of the BP-Scale (NT=30 cents)

|             |             |             |             |              |             |             |             |             |             |             |             |              |             |
|-------------|-------------|-------------|-------------|--------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|--------------|-------------|
| 1           | 2           | 3           | 4           | 5            | 6           | 7           | 8           | 9           | 10          | 11          | 12          | 13           | 14          |
| 1:1         | 32:35       | 27:32       | 7:9         | 32:45        | 21:32       | 3:5         | 5:9         | 1:2         | 15:32       | 3:7         | 32:81       | 9:25         | 1:3         |
| 0           | 155         | 294         | 435         | 590          | 729         | 884         | 1018        | 1200        | 1312        | 1467        | 1608        | 1769         | 1902        |
| [0]         | [146]       | [293]       | [439]       | [585]        | [732]       | [878]       | [1024]      | [1170]      | [1317]      | [1463]      | [1609]      | [1756]       | [1902]      |
| 0           | +9          | +1          | -4          | +5           | -3          | +6          | -6          | +30         | -5          | +4          | -1          | +13          | 0           |
| C $\flat_0$ | D $\flat_0$ | E $\flat_0$ | E $\flat_0$ | F $\sharp_0$ | G $\flat_0$ | A $\flat_0$ | B $\flat_0$ | C $\flat_1$ | D $\flat_1$ | E $\flat_1$ | E $\flat_1$ | F $\sharp_1$ | G $\flat_1$ |



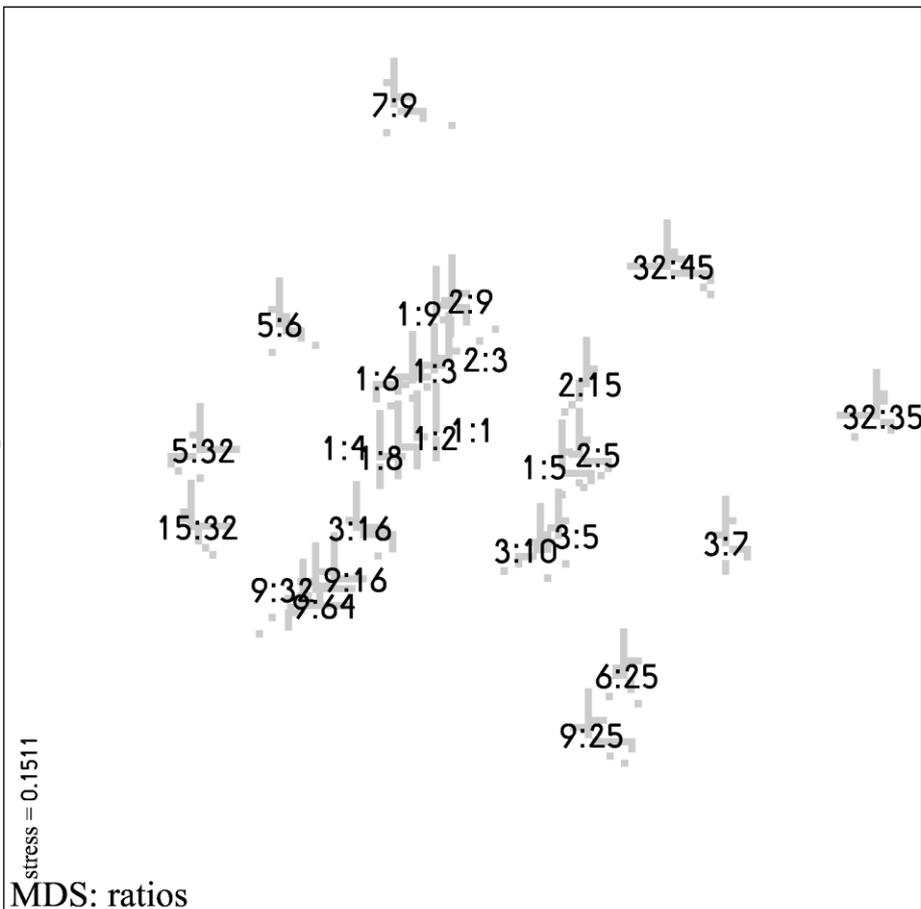
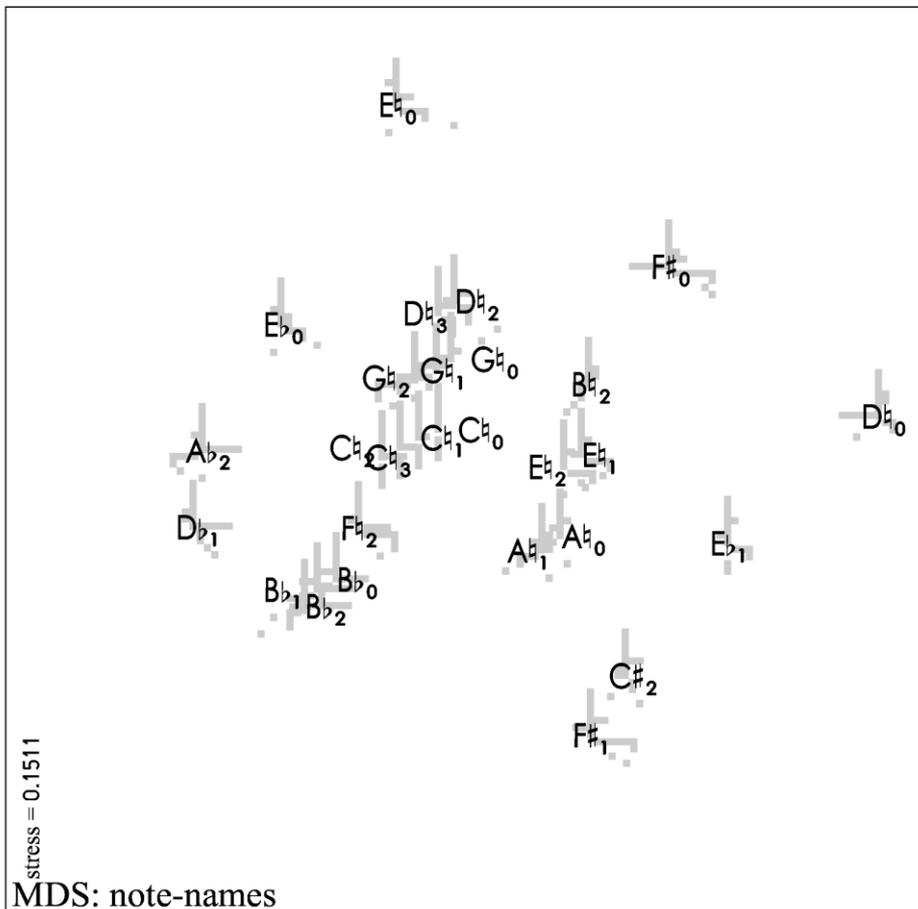
# MDS of a rationalization of the BP-Scale (NT=29 cents)

|                  |                  |                  |                  |                  |                  |                  |                  |                  |                  |                  |                  |                  |                  |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| 1                | 2                | 3                | 4                | 5                | 6                | 7                | 8                | 9                | 10               | 11               | 12               | 13               | 14               |
| 1:1              | 32:35            | 27:32            | 7:9              | 32:45            | 21:32            | 3:5              | 5:9              | 32:63            | 15:32            | 3:7              | 32:81            | 9:25             | 1:3              |
| 0                | 155              | 294              | 435              | 590              | 729              | 884              | 1018             | 1173             | 1312             | 1467             | 1608             | 1769             | 1902             |
| cents            | [146]            | [293]            | [439]            | [585]            | [732]            | [878]            | [1024]           | [1170]           | [1317]           | [1463]           | [1609]           | [1756]           | [1902]           |
| 0                | +9               | +1               | -4               | +5               | -3               | +6               | -6               | +3               | -5               | +4               | -1               | +13              | 0                |
| C <sub>h</sub> 0 | D <sub>h</sub> 0 | E <sub>b</sub> 0 | E <sub>h</sub> 0 | F <sub>#</sub> 0 | G <sub>h</sub> 0 | A <sub>h</sub> 0 | B <sub>b</sub> 0 | C <sub>h</sub> 0 | D <sub>b</sub> 1 | E <sub>b</sub> 1 | E <sub>h</sub> 1 | F <sub>#</sub> 1 | G <sub>h</sub> 1 |

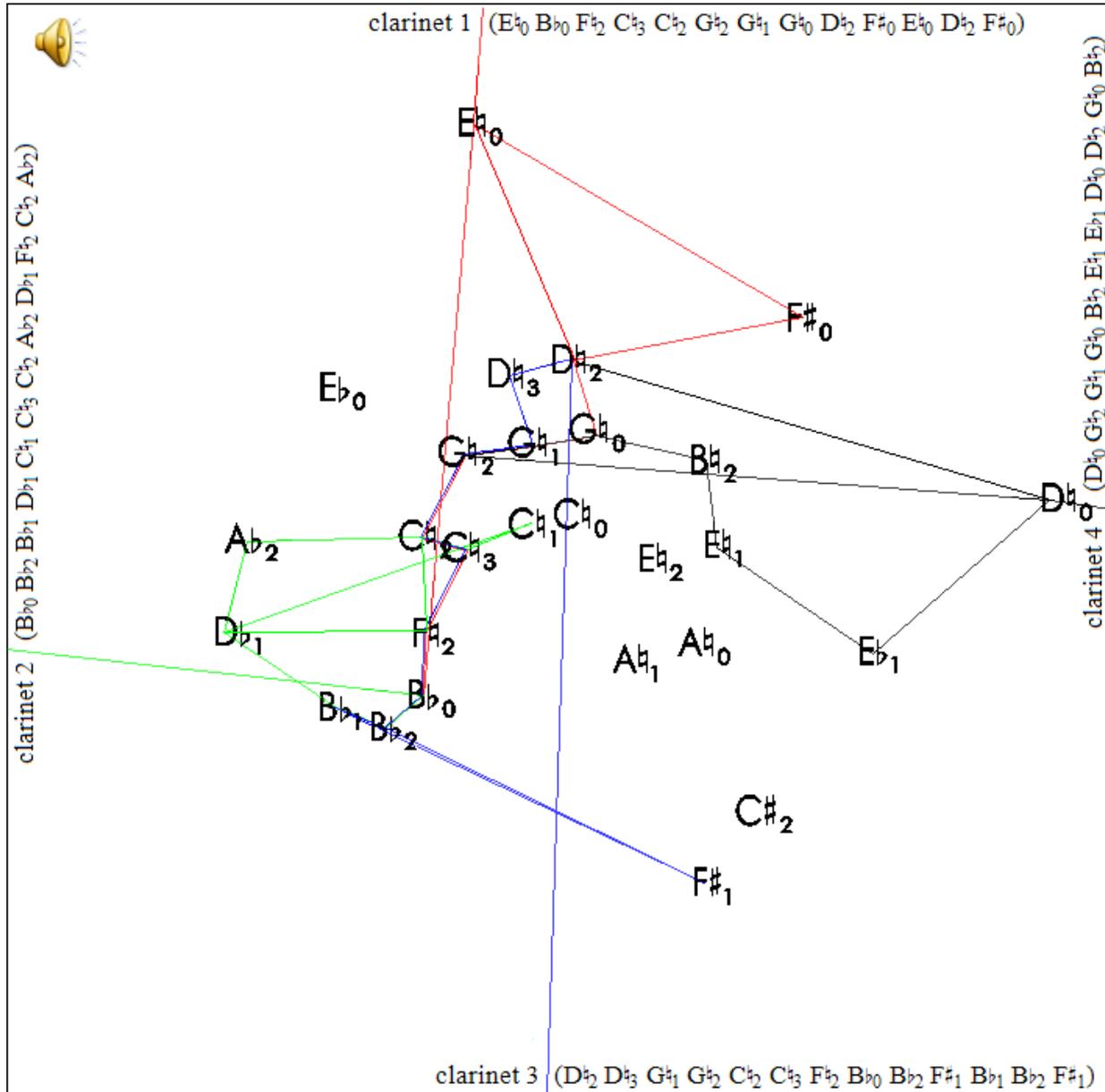


# MDS of a rationalization of a two-frame BP-Scale

|                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |                             |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 1                           | 2                           | 3                           | 4                           | 5                           | 6                           | 7                           | 8                           | 9                           | 10                          | 11                          | 12                          | 13                          | 14                          | 15                          | 16                          | 17                          | 18                          | 19                          | 20                          | 21                          | 22                          | 23                          | 24                          | 25                          | 26                          | 27                          |                             |
| 1:1                         | 32:35                       | 5:6                         | 7:9                         | 32:45                       | 2:3                         | 3:5                         | 9:16                        | 1:2                         | 15:32                       | 3:7                         | 2:5                         | 9:25                        | 1:3                         | 3:10                        | 9:32                        | 1:4                         | 6:25                        | 2:9                         | 1:5                         | 3:16                        | 1:6                         | 5:32                        | 9:64                        | 2:15                        | 1:8                         | 1:9                         |                             |
| cents                       | 0                           | 155                         | 316                         | 435                         | 590                         | 702                         | 884                         | 996                         | 1200                        | 1312                        | 1467                        | 1586                        | 1769                        | 1902                        | 2084                        | 2196                        | 2400                        | 2471                        | 2604                        | 2786                        | 2898                        | 3102                        | 3214                        | 3396                        | 3488                        | 3600                        | 3804                        |
| [0]                         | [146]                       | [293]                       | [439]                       | [585]                       | [732]                       | [878]                       | [1024]                      | [1170]                      | [1317]                      | [1463]                      | [1609]                      | [1756]                      | [1902]                      | [2048]                      | [2195]                      | [2341]                      | [2487]                      | [2634]                      | [2780]                      | [2926]                      | [3072]                      | [3219]                      | [3365]                      | [3511]                      | [3658]                      | [3804]                      |                             |
| 0                           | +9                          | +23                         | -4                          | +5                          | -30                         | +6                          | -28                         | +30                         | -5                          | +4                          | -23                         | +13                         | 0                           | +36                         | +1                          | +59                         | -16                         | -30                         | +6                          | -28                         | +30                         | -5                          | +31                         | -23                         | -58                         | 0                           |                             |
| C <sub>h</sub> <sub>0</sub> | D <sub>h</sub> <sub>0</sub> | E <sub>h</sub> <sub>0</sub> | E <sub>h</sub> <sub>0</sub> | F <sub>h</sub> <sub>0</sub> | G <sub>h</sub> <sub>0</sub> | A <sub>h</sub> <sub>0</sub> | B <sub>h</sub> <sub>0</sub> | C <sub>h</sub> <sub>1</sub> | D <sub>h</sub> <sub>1</sub> | E <sub>h</sub> <sub>1</sub> | E <sub>h</sub> <sub>1</sub> | F <sub>h</sub> <sub>1</sub> | G <sub>h</sub> <sub>1</sub> | A <sub>h</sub> <sub>1</sub> | B <sub>h</sub> <sub>1</sub> | C <sub>h</sub> <sub>2</sub> | C <sub>h</sub> <sub>2</sub> | D <sub>h</sub> <sub>2</sub> | E <sub>h</sub> <sub>2</sub> | F <sub>h</sub> <sub>2</sub> | G <sub>h</sub> <sub>2</sub> | A <sub>h</sub> <sub>2</sub> | B <sub>h</sub> <sub>2</sub> | B <sub>h</sub> <sub>2</sub> | B <sub>h</sub> <sub>2</sub> | C <sub>h</sub> <sub>3</sub> | D <sub>h</sub> <sub>3</sub> |



# The previous MDS as a starting point for *Pinball Play*



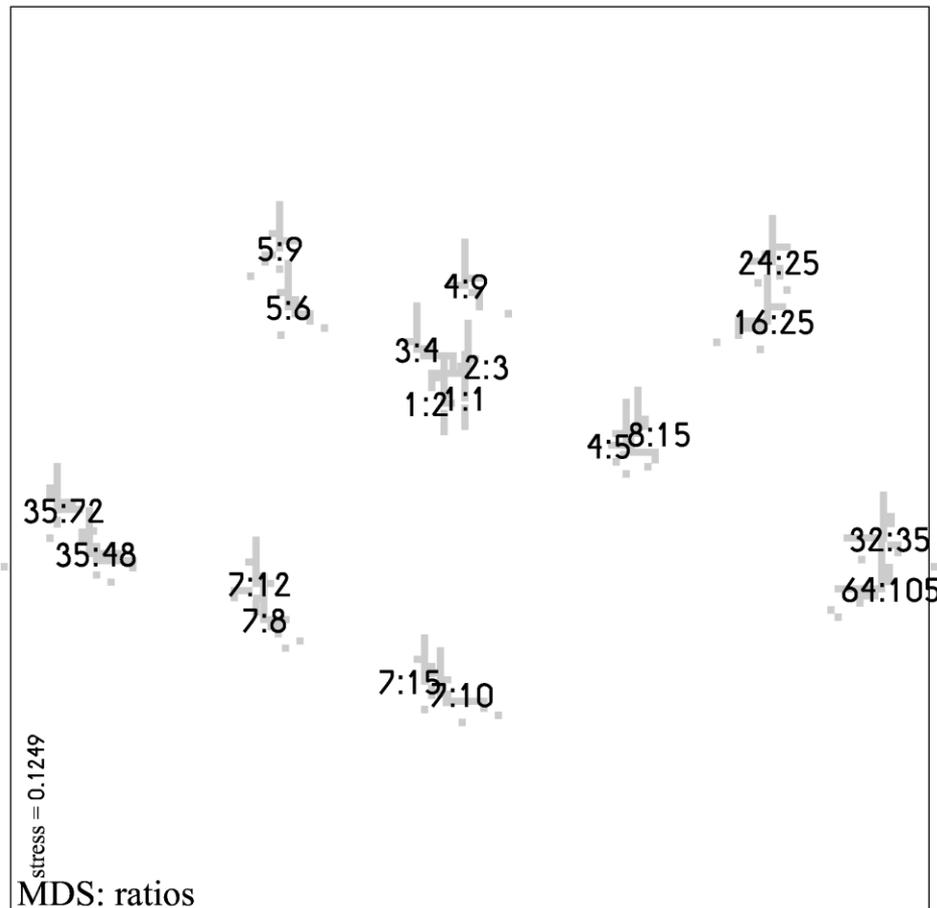
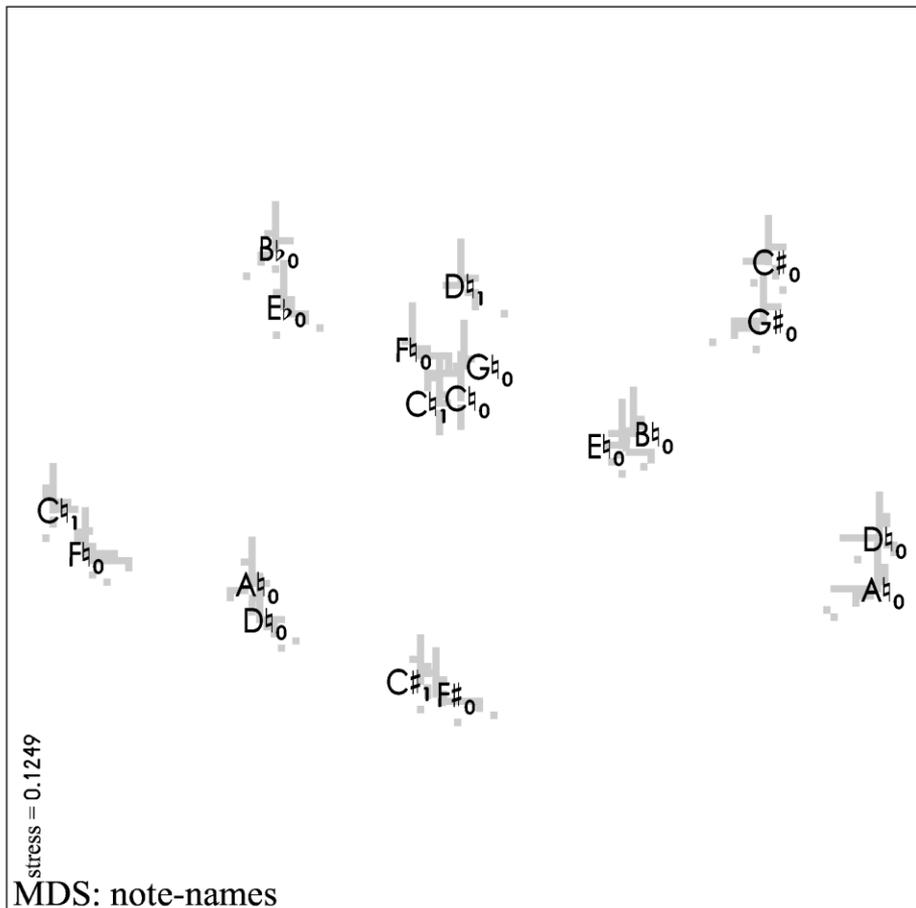
# Cogency\* of rationalized equal-tempered scales: 6-24 semitones divided into 5-24 equal parts

|              | divisions: 5 | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  | 21  | 22  | 23  | 24  |  |
|--------------|--------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|--|
| semitones: 6 | 159          | 317 | 253 | 172 | 255 | 272 | 274 | 270 |     |     |     |     |     |     |     |     |     |     |     |     |  |
| 7            | 136          | 178 | 395 | 231 | 652 | 295 | 331 | 485 | 325 | 317 |     |     |     |     |     |     |     |     |     |     |  |
| 8            | 166          | 179 | 196 | 408 | 194 | 233 | 224 | 250 | 296 | 236 | 294 | 312 |     |     |     |     |     |     |     |     |  |
| 9            | 267          | 149 | 314 | 194 | 350 | 208 | 255 | 194 | 266 | 325 | 230 | 342 | 325 | 330 |     |     |     |     |     |     |  |
| 10           | 357          | 247 | 191 | 221 | 209 | 422 | 220 | 257 | 285 | 406 | 281 | 370 | 489 | 292 | 335 | 354 |     |     |     |     |  |
| 11           | 148          | 231 | 172 | 193 | 248 | 225 | 388 | 231 | 244 | 283 | 309 | 306 | 334 | 276 | 397 | 369 | 369 | 353 |     |     |  |
| 12           | 345          | 453 | 295 | 177 | 209 | 256 | 435 | 440 | 230 | 330 | 316 | 219 | 393 | 328 | 369 | 291 | 308 | 422 | 496 | 390 |  |
| 13           |              | 157 | 142 | 200 | 218 | 234 | 237 | 336 | 450 | 236 | 306 | 263 | 294 | 258 | 263 | 268 | 314 | 361 | 381 | 390 |  |
| 14           |              | 397 | 552 | 313 | 198 | 171 | 211 | 214 | 241 | 441 | 247 | 289 | 255 | 388 | 286 | 382 | 373 | 443 | 356 | 425 |  |
| 15           |              | 261 | 160 | 151 | 307 | 195 | 214 | 203 | 238 | 234 | 479 | 262 | 248 | 264 | 308 | 262 | 328 | 298 | 273 | 460 |  |
| 16           |              |     | 280 | 566 | 332 | 167 | 263 | 234 | 266 | 190 | 239 | 485 | 271 | 208 | 269 | 367 | 328 | 257 | 347 | 367 |  |
| 17           |              |     | 340 | 177 | 178 | 341 | 236 | 200 | 244 | 302 | 230 | 228 | 490 | 266 | 246 | 320 | 327 | 348 | 314 | 468 |  |
| 18           |              |     |     | 216 | 596 | 268 | 235 | 201 | 222 | 232 | 284 | 249 | 229 | 491 | 264 | 281 | 385 | 312 | 303 | 262 |  |
| 19           |              |     |     | 413 | 198 | 215 | 377 | 216 | 287 | 243 | 251 | 295 | 270 | 245 | 562 | 245 | 273 | 299 | 284 | 349 |  |
| 20           |              |     |     | 252 | 170 | 569 | 189 | 329 | 214 | 220 | 258 | 226 | 268 | 263 | 271 | 532 | 264 | 277 | 346 | 326 |  |
| 21           |              |     |     |     | 374 | 211 | 223 | 364 | 198 | 216 | 227 | 265 | 264 | 255 | 251 | 265 | 536 | 266 | 343 | 361 |  |
| 22           |              |     |     |     | 353 | 174 | 554 | 187 | 361 | 252 | 271 | 262 | 284 | 339 | 268 | 294 | 259 | 560 | 274 | 306 |  |
| 23           |              |     |     |     |     | 305 | 220 | 222 | 371 | 234 | 233 | 250 | 274 | 262 | 324 | 257 | 362 | 269 | 563 | 279 |  |
| 24           |              |     |     |     |     | 463 | 239 | 617 | 192 | 400 | 203 | 238 | 197 | 286 | 247 | 346 | 250 | 374 | 256 | 597 |  |

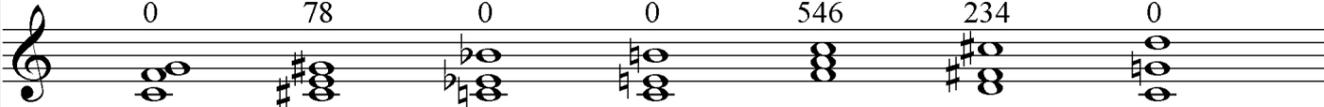
\*  $10000 \times \text{Specific Harmonicity} \times \frac{1}{2}\text{-step frame-size}^{0.45} / \text{RMS cent deviation}$

# MDS of a rationalization of a 2-frame Carlos- $\alpha$ Scale (perfect fifth divided into nine equal parts)

|          |           |          |          |          |          |          |          |           |          |           |          |          |          |          |          |          |           |          |
|----------|-----------|----------|----------|----------|----------|----------|----------|-----------|----------|-----------|----------|----------|----------|----------|----------|----------|-----------|----------|
| 1        | 2         | 3        | 4        | 5        | 6        | 7        | 8        | 9         | 10       | 11        | 12       | 13       | 14       | 15       | 16       | 17       | 18        | 19       |
| 1:1      | 24:25     | 32:35    | 7:8      | 5:6      | 4:5      | 3:4      | 35:48    | 7:10      | 2:3      | 16:25     | 64:105   | 7:12     | 5:9      | 8:15     | 1:2      | 35:72    | 7:15      | 4:9      |
| 0        | 71        | 155      | 231      | 316      | 386      | 498      | 547      | 617       | 702      | 773       | 857      | 933      | 1018     | 1088     | 1200     | 1249     | 1319      | 1404     |
| [0]      | [78]      | [156]    | [234]    | [312]    | [390]    | [468]    | [546]    | [624]     | [702]    | [780]     | [858]    | [936]    | [1014]   | [1092]   | [1170]   | [1248]   | [1326]    | [1404]   |
| 0        | -7        | -1       | -3       | +4       | -4       | +30      | +1       | -7        | 0        | -7        | -1       | -3       | +4       | -4       | +30      | +1       | -7        | 0        |
| C $^b_0$ | C $^\#_0$ | D $^b_0$ | D $^b_0$ | E $^b_0$ | E $^b_0$ | F $^b_0$ | F $^b_0$ | F $^\#_0$ | G $^b_0$ | G $^\#_0$ | A $^b_0$ | A $^b_0$ | B $^b_0$ | B $^b_0$ | C $^b_1$ | C $^b_1$ | C $^\#_1$ | D $^b_1$ |



# Triads derivable from the previous slide

| Tempered Intervals in cents                                                        |       |      |      |       |      |      |
|------------------------------------------------------------------------------------|-------|------|------|-------|------|------|
| 702                                                                                | 780   | 1014 | 1092 | 1248  | 1326 | 1404 |
| 468                                                                                | 390   | 312  | 390  | 936   | 624  | 702  |
| 0                                                                                  | 78    | 0    | 0    | 546   | 234  | 0    |
|  |       |      |      |       |      |      |
| 2:3                                                                                | 16:25 | 5:9  | 8:15 | 35:72 | 7:15 | 4:9  |
| 3:4                                                                                | 4:5   | 5:6  | 4:5  | 7:12  | 7:10 | 2:3  |
| 1:1                                                                                | 24:25 | 1:1  | 1:1  | 35:48 | 7:8  | 1:1  |
| Rationalized Intervals                                                             |       |      |      |       |      |      |

